

Learning Goals

- Basic definitions of finite probabilities: sample space, probability, events
- State and apply union bound.
- Define independence, and apply its properties in probability calculations
- Contention resolution with random access, and analysis of its efficiency

Borges's Garden of Forking Paths



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- A probability space is defined by weights on those realizations.

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- If everything is fair, then each outcome has probability mass $1/36$.
- Let \mathcal{E} be the event that the sum of the two numbers is 11, then $\mathcal{E} = \{(6, 5), (5, 6)\}$, so $\Pr[\mathcal{E}] = 1/18$.

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Exercise: If A and B are independent, then so are \bar{A} and B , and so are \bar{A} and \bar{B} .

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 - 1 \forall “measurable” event A , $\Pr[A] \geq 0$.
 - 2 $\Pr[\Omega] = 1$.
 - 3 for *countably* many disjoint events A_1, A_2, \dots , $\Pr[\sqcup_i A_i] = \sum_i \Pr[A_i]$.

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 - 3 for *countably* many disjoint events A_1, A_2, \dots , $\Pr[\sqcup_i A_i] = \sum_i \Pr[A_i]$.
- It takes measure theory to make things rigorous. We will make use of such probability spaces in very few occasions in this course.

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- Trivial if the tasks can agree on some ordering and requests the service one by one.
- Problem: The tasks cannot talk with each other and there is no central authority.
- **Randomized strategy:** In each time step, each task requests with some small probability p , *independently*.

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- Let $S[i, t]$ denote the event that task i sends a request at time t *and* gets served, then

$$\Pr[S[i, t]] = \Pr \left[A[i, t] \cap \bigcap_{j \neq i} \overline{A[j, t]} \right] = p(1 - p)^{n-1}.$$

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- To maximize $\Pr[S[i, t]]$, set $p = 1/n$.

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- 1 The function $(1 - \frac{1}{n})^n$ converges monotonically from $\frac{1}{4}$ up to $\frac{1}{e}$ as n increases from 2.
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So $1/(en) \leq \Pr[S[i, t]] \leq 1/(2n)$. Therefore $\Pr[S[i, t]]$ is asymptotically $\Theta(1/n)$.

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- Probability with which task i does not succeed in the first t steps:

$$\Pr \left[\bigcap_{r=1}^t \overline{S[i, r]} \right] = \prod_{r=1}^t [1 - \Pr[S[i, r]]] = \left[1 - \frac{1}{n} \left(1 - \frac{1}{n} \right)^{n-1} \right]^t.$$

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- Big picture (useful rough estimations): if we have a biased coin that gives Heads with probability $1/k$:
 - In about k independent tosses, one “expects” to see a Heads;
 - However, with constant probability, a Heads doesn't show in k tosses;
 - But if one tosses the coin $\Theta(k \log k)$ times, the probability that no Heads shows up quickly tends to 0.

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- Let $F[i, t]$ denote the event that task i fails in the first t steps, we have shown $\Pr[F[i, t]] \leq e^{-t/en} \leq n^{-c}$ for $t = \lceil en \cdot c \ln n \rceil$.

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By the union bound, we have

$$\Pr \left[\bigcup_{i=1}^n F[i, t] \right] \leq \sum_{i=1}^n e^{-t/en} = ne^{-\frac{t}{en}}.$$

So for $t = \lceil 2en \ln n \rceil$, this is at most $\frac{1}{n}$.

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- A useful upper bound: for $x \in (0, 1)$, $1 - x < e^{-x}$. So the above probability is at most $\prod_{i=1}^{n-1} e^{-i/365} = e^{-n(n-1)/730}$.
- As long as $e^{-n(n-1)/730} < \frac{1}{2}$, i.e., $n \geq 23$, you should bet that some pair of students have the same birthday.