Learning Goals

- Random variables and their expectations
- Expectation of some distributions (Indicator variables/Bernoulli, binomial, geometric)
- Linearity of expectations
- Analyze two examples: guessing cards and coupon collection

Random variables

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- Example: For an event A, let X be 1 if A happens, and 0 if not. Then $\Pr[X = 1] = \Pr[A]$.
  - X is called the *indicator variable* of A.
  - A random variable that only takes values 0 or 1 is said to be drawn from a *Bernoulli distribution*.

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- Example: If X is the result of a die toss, then

$$\mathrm{E}[X] = \frac{1}{6}\sum_{i=1}^{6} i = \frac{7}{2}.$$

$$\mathrm{E}\left[X^2\right] = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{91}{6}.$$

Note $\mathrm{E}[X^2] \neq (\mathrm{E}[X])^2$.

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- A random variable X is *geometrically distributed* with parameter $p \in (0,1)$ if X takes values in $\mathbb{N}$ and $\Pr[X = k] = (1-p)^{k-1}p$.

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Theorem

For any collection of random variables $X_1, \cdots, X_n$ (defined on the same probability space),

$$\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}\left[X_i\right].$$

Independence among random variables

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Remark

Linearity of expectation does NOT need independence among the random variables!

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Shuffle a deck of n distinct cards, and reveal them one by one. Before each revelation, make a uniformly random guess among the unrevealed cards. How many guesses are correct in expectation?

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- Let X_i be the indicator variable for the i-th guess being correct, then $\mathrm{E}[X_i] = 1/n$.
- The total number of correct guesses is $X := \sum_{i=1}^{n} X_i$. So $\mathrm{E}[X] = \sum_{i=1}^{n} \mathrm{E}[X_i] = n \cdot \frac{1}{n} = 1$.

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- The total number of crrect guess is $Y := \sum_i Y_i$. So

$$\mathrm{E}[Y] = \sum_{i=1}^{n} \mathrm{E}[Y_i] = \sum_{i=1}^{n} \frac{1}{n-i+1} = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n.$$

Examples of linearity of expectations: Coupon collection

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- There are $n-i+1$ unseen coupons, and the probability we see one of them in each purchase is $\frac{n-i+1}{n}$.
- $\mathrm{E}[X_i] = \frac{n}{n-i+1}$ (from the earlier example about tossing coins.)
- Therefore the expected total number of purchases is

$$\sum_{i=1}^{n} \frac{n}{n-i+1} = n \cdot \sum_{i=1}^{n} \frac{1}{i} \approx n \ln n.$$

Recipe for expectation calculation

- Express the quantity we are interested in as a random variable
- Express the random variable as a sum of random variables whose expectations are easy to compute
- Apply linearity of expectation (without worrying about independence)!