Learning Goal

- Reduction from Baseball Elimination to max flow
- Interpreting min cuts in Baseball Elimination

Motivating Problem

- There is a set S of baseball teams
- Each team $x \in S$ has won w_x games in the past;

Motivating Problem

- There is a set S of baseball teams
- Each team $x \in S$ has won w_x games in the past;
- Between each pair of teams x and y, there are still g_{xy} games to be played; (g_{xy} may be 0)

Motivating Problem

- There is a set S of baseball teams
- Each team $x \in S$ has won w_x games in the past;
- Between each pair of teams x and y, there are still g_{xy} games to be played; (g_{xy} may be 0)
- Question: is it still possible for a particular team z to be a champion, i.e., its number of games won is the highest (allowing ties) after all games are played?

An example

- Obvious first step: for best case scenario, suppose team z wins all future games.

An example

- Obvious first step: for best case scenario, suppose team z wins all future games.
- Example:
  - Past games won:

    New York: 92; Baltimore: 91; Toronto: 91; Boston: 90.
  - One game left between every two teams except between New York and Boston.

An example

- Obvious first step: for best case scenario, suppose team z wins all future games.
- Example:
  - Past games won:

    New York: 92; Baltimore: 91; Toronto: 91; Boston: 90.
  - One game left between every two teams except between New York and Boston.
  - Consider z = Boston. In best case scenario, Boston wins both games left, totaling 92 wins.

An example

- Obvious first step: for best case scenario, suppose team z wins all future games.
- Example:
  - Past games won:

    New York: 92; Baltimore: 91; Toronto: 91; Boston: 90.
  - One game left between every two teams except between New York and Boston.
  - Consider $z =$ Boston. In best case scenario, Boston wins both games left, totaling 92 wins.
  - In order for Boston to be the highest, New York must lose both games, which means Baltimore and Toronto are both at 92 wins before counting the game between them;

An example

- Obvious first step: for best case scenario, suppose team z wins all future games.
- Example:
  - Past games won:

    New York: 92; Baltimore: 91; Toronto: 91; Boston: 90.
  - One game left between every two teams except between New York and Boston.
  - Consider $z =$ Boston. In best case scenario, Boston wins both games left, totaling 92 wins.
  - In order for Boston to be the highest, New York must lose both games, which means Baltimore and Toronto are both at 92 wins before counting the game between them;
  - But then counting the game between Baltimore and Toronto, one of them has 93 wins. Therefore Boston already cannot be a champion.

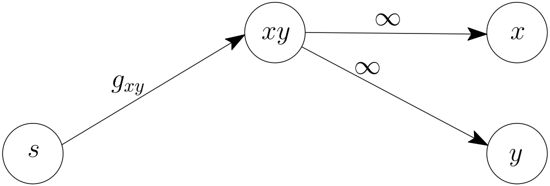
Connection to Flows

- Assume z wins all future games, totaling m wins, the remaining question is whether the remaining games can be played so that all other teams have $\leq m$ wins.

Connection to Flows

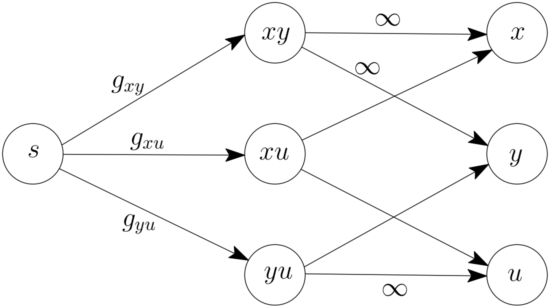
- Assume z wins all future games, totaling m wins, the remaining question is whether the remaining games can be played so that all other teams have $\leq m$ wins.
- Key idea:
  - Deciding the outcome of a game is like allocating a resource between the two teams involved;
  - Each team having no more than m wins is like an upper bound on its total allocation.

Construction of the flow network: Step 1



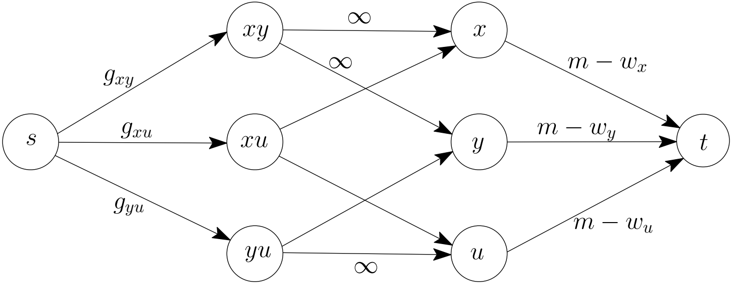
Allocating g_{xy} resources (wins) between team x and y.

Construction of the flow network: Step 2



Allocating wins between every pair of teams.

Construction of the flow network: Step 3



Capping the total wins allocated to each team.

Reduction to Max Flow

- Construct flow network G:
  - Add source s and sink t;
  - for each pair of teams $x, y \neq z$, create node u_{xy};
  - for each team $x \neq z$ create node v_x;

Reduction to Max Flow

- Construct flow network G:
  - Add source s and sink t;
  - for each pair of teams $x, y \neq z$, create node u_{xy};
  - for each team $x \neq z$ create node v_x;
  - For each u_{xy}, add edge from s to u_{xy} with capacity g_{xy};

Reduction to Max Flow

- Construct flow network G:
  - Add source s and sink t;
  - for each pair of teams $x, y \neq z$, create node u_{xy};
  - for each team $x \neq z$ create node v_x;
  - For each u_{xy}, add edge from s to u_{xy} with capacity g_{xy};
  - For each v_x, add edge from v_x to t with capacity $m - w_x$;

Reduction to Max Flow

- Construct flow network G:
  - Add source s and sink t;
  - for each pair of teams $x, y \neq z$, create node u_{xy};
  - for each team $x \neq z$ create node v_x;
  - For each u_{xy}, add edge from s to u_{xy} with capacity g_{xy};
  - For each v_x, add edge from v_x to t with capacity $m - w_x$;
  - For each u_{xy}, add edges from u_{xy} to v_x and v_y, with infinite capacity.

Claim

Team z can be a champion if and only if G has a flow that saturates all edges from s.

Reduction to Max Flow

- Construct flow network G:
  - Add source s and sink t;
  - for each pair of teams $x, y \neq z$, create node u_{xy};
  - for each team $x \neq z$ create node v_x;
  - For each u_{xy}, add edge from s to u_{xy} with capacity g_{xy};
  - For each v_x, add edge from v_x to t with capacity $m - w_x$;
  - For each u_{xy}, add edges from u_{xy} to v_x and v_y, with infinite capacity.

Claim

Team z can be a champion if and only if G has a flow that saturates all edges from s.

Proof.

1. Any scenario with z as a champion corresponds to such a flow;
2. Any such flow corresponds to a scenario with z as a champion.

□

Interpreting Consequences of Max Flow Min Cut

- Corollary of Max Flow Min Cut Theorem: A max flow saturates edges from s if and only if the cut with $\{s\}$ on one side is a min cut in G.

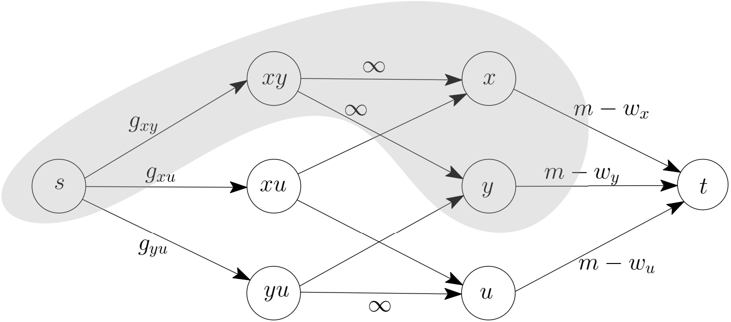
Interpreting Consequences of Max Flow Min Cut

- Corollary of Max Flow Min Cut Theorem: A max flow saturates edges from s if and only if the cut with $\{s\}$ on one side is a min cut in G.
- What happens when z is eliminated? There must be a cut with capacity $< \sum_{\{x,y\}} g_{xy}$.

Interpreting Consequences of Max Flow Min Cut

- Corollary of Max Flow Min Cut Theorem: A max flow saturates edges from s if and only if the cut with $\{s\}$ on one side is a min cut in G.
- What happens when z is eliminated? There must be a cut with capacity $< \sum_{\{x,y\}} g_{xy}$.
- What is the capacity of such an s-t cut (A, B)?

Illustration of a cut



Capacity of the cut: $g_{xu} + g_{yu} + (m - w_x) + (m - w_y)$.

Characterizing min cuts

- Consider any s-t cut (A, B) of G.
- Let U be the set of nodes for games, and V the set of nodes for teams.

Characterizing min cuts

- Consider any s-t cut (A, B) of G.
- Let U be the set of nodes for games, and V the set of nodes for teams.
- Let T be the set of teams on the side of the sink in the cut. When can (A, B) be possibly a min cut?

Characterizing min cuts

- Consider any s-t cut (A, B) of G.
- Let U be the set of nodes for games, and V the set of nodes for teams.
- Let T be the set of teams on the side of the sink in the cut. When can (A, B) be possibly a min cut?
  1. If any game that involves a team in T is not also in B, $c(A, B) = \infty$.

Characterizing min cuts

- Consider any s-t cut (A, B) of G.
- Let U be the set of nodes for games, and V the set of nodes for teams.
- Let T be the set of teams on the side of the sink in the cut. When can (A, B) be possibly a min cut?
  1. If any game that involves a team in T is not also in B, $c(A, B) = \infty$.
  2. If a game that doesn't involve any team in T is in B, moving it to A reduces the cut's capacity.

Characterizing min cuts

- Consider any s-t cut (A, B) of G.
- Let U be the set of nodes for games, and V the set of nodes for teams.
- Let T be the set of teams on the side of the sink in the cut. When can (A, B) be possibly a min cut?
  1. If any game that involves a team in T is not also in B, $c(A, B) = \infty$.
  2. If a game that doesn't involve any team in T is in B, moving it to A reduces the cut's capacity.
- Therefore, fixing T, it minimizes the cut's capacity to let $B \cap U$ be the set of games involving any team in T.

Interpreting min cuts

- The capacity of such a cut (A, B) is

$$c(A,B) = \sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy}.$$

Interpreting min cuts

- The capacity of such a cut (A, B) is

$$c(A, B) = \sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy}.$$

- By the previous discussion, there exists an s-t cut (A, B) with $c(A, B) < \sum_{\{x,y\}} g_{xy}$.

Interpreting min cuts

- The capacity of such a cut (A, B) is

$$c(A, B) = \sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy}.$$

- By the previous discussion, there exists an s-t cut (A, B) with $c(A, B) < \sum_{\{x,y\}} g_{xy}$.
- Hence there exists a set of teams T with

$$\sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy} < \sum_{\{x,y\}} g_{xy}.$$

Interpreting min cuts

- The capacity of such a cut (A, B) is

$$c(A,B) = \sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy}.$$

- By the previous discussion, there exists an s-t cut (A, B) with $c(A,B) < \sum_{\{x,y\}} g_{xy}$.
- Hence there exists a set of teams T with

$$\sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy} < \sum_{\{x,y\}} g_{xy}.$$

$$\Rightarrow \sum_{\{x,y\} \subseteq S \setminus T} g_{xy} > \sum_{x \in S \setminus T} (m - w_x).$$

Interpreting min cuts

- The capacity of such a cut (A, B) is

$$c(A,B) = \sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy}.$$

- By the previous discussion, there exists an s-t cut (A, B) with $c(A,B) < \sum_{\{x,y\}} g_{xy}$.
- Hence there exists a set of teams T with

$$\sum_{x \notin T} (m - w_x) + \sum_{\{x,y\} \not\subseteq S \setminus T} g_{xy} < \sum_{\{x,y\}} g_{xy}.$$

$$\Rightarrow \sum_{\{x,y\} \subseteq S \setminus T} g_{xy} > \sum_{x \in S \setminus T} (m - w_x).$$

- The total number of games among teams in $S \setminus T$ exceeds the sum of upper bound of games each of them can win in order not to beat team z.