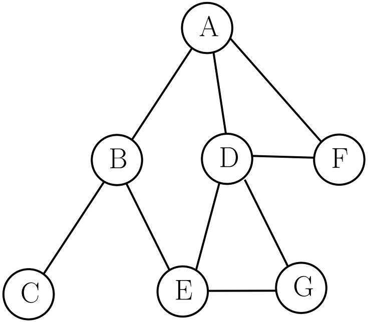
Learning Goals

- Review steps of breadth first search (BFS) and depth first search (DFS) algorithms
- Running time of BFS and DFS
- Properties of BFS and DFS trees

Graph Traversal

Problem: Given an undirected graph $G = (V, E)$ and two nodes $s, t \in V$, decide whether there is a path connecting s and t.



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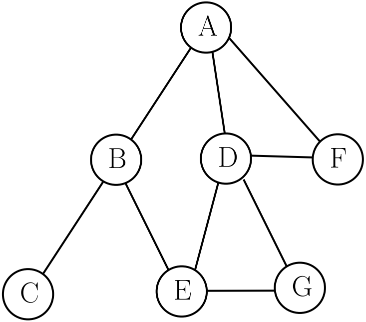
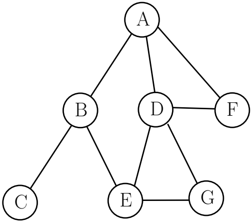


- Breadth First Search (BFS): mark s as visited, imeediately mark all neighbors of s as visited, and THEN recursively do the same for all the nodes that are newly marked as visited.

BFS example



The given graph

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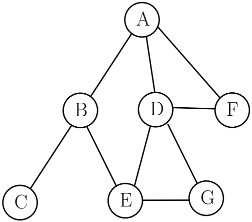

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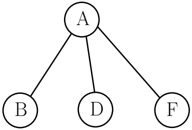

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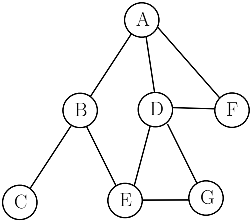

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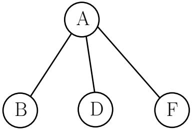

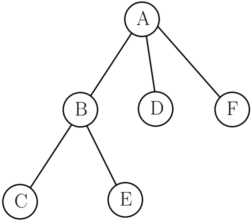

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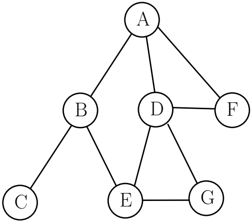

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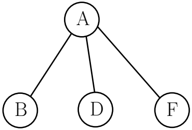

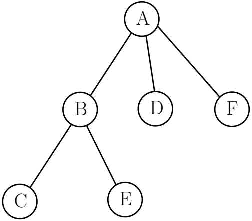

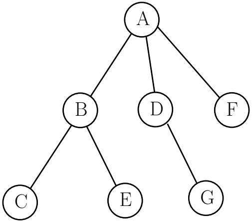

The BFS tree

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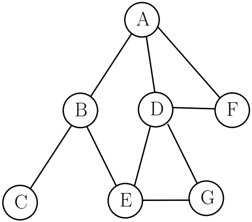

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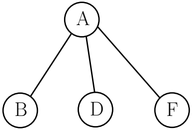

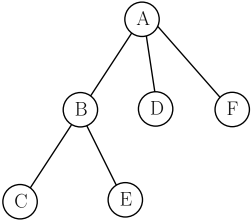

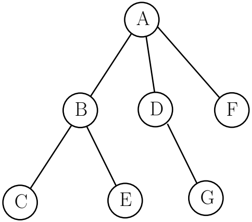

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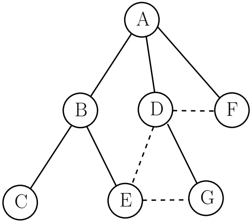

BFS tree with edges in the original graph.

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  - This is called *linear time*.

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Definition (Layers of a BFS tree)

The first layer, L_1, of a BFS tree is the singleton set of the starting node s; then the $(i+1)^{\text{-st}}$ layer L_{i+1} is the set of nodes that are newly marked visited when the algorithm is processing a node in L_i.

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The shortest path from s to any node in L_i has length $i-1$ (i.e., has $i-1$ edges).

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For any $(u,v) \in E$, in a BFS tree if u is in L_i and v in L_j, then $|i-j| \leq 1$.

Depth First Search

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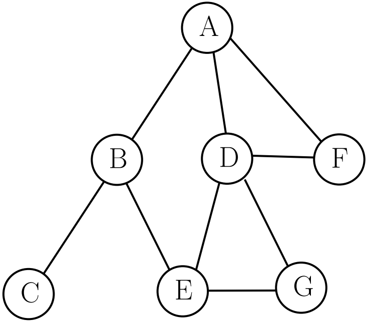

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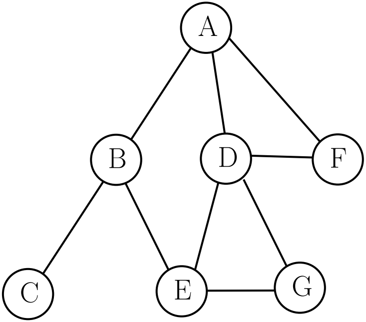

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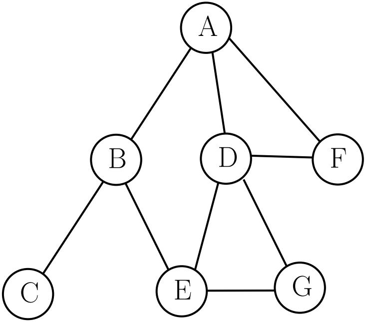

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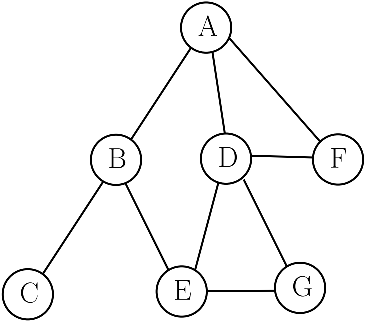
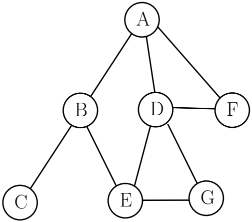


- Depth First Search (DFS) idea: go headlong till a dead end, then backtrack.
- Mark s as visited, then for each neighbor of s, if it is unmarked, recursively do the same.
- In other words, process the first neighbor of s completely before going on to the next unvisited neighbor.

DFS example



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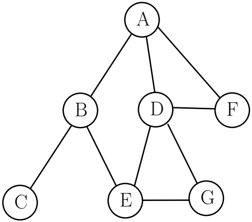

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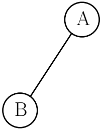

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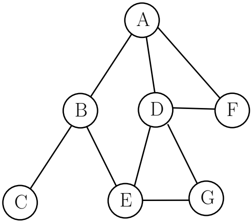

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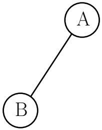

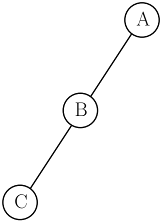

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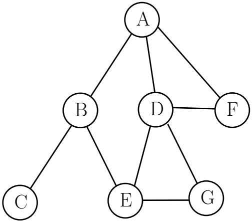

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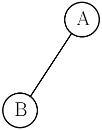

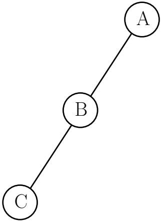

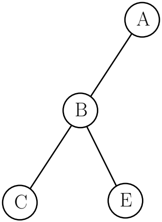

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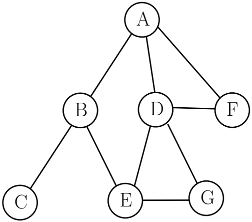

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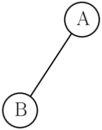

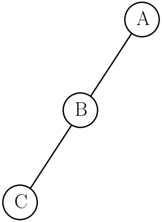

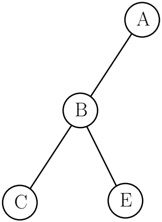

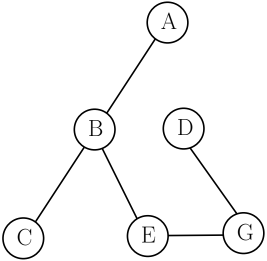

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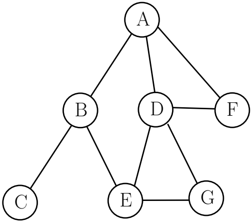

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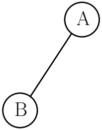

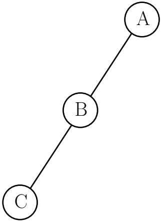

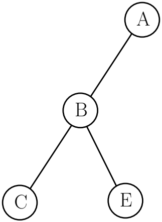

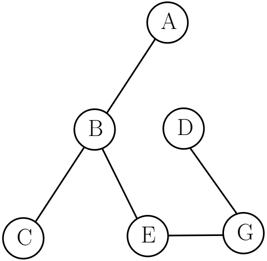

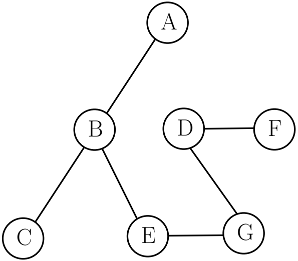

A DFS tree

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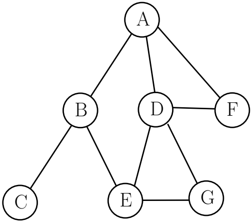

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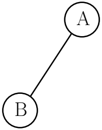

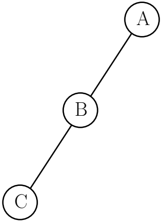

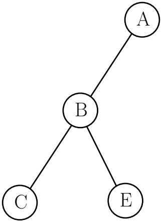

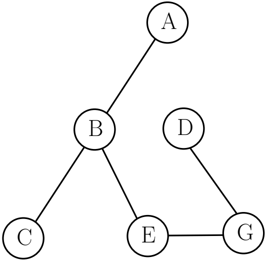

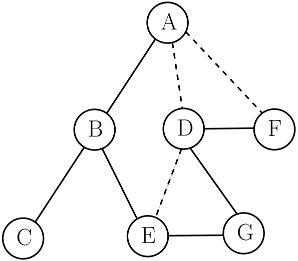

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- DFS also visits all the nodes in the same connected component as s.
- Running time: $O(n+m)$.
  - Each edge is followed at most twice; each "stacking" follows from an edge visit.

More on DFS

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Proof.

Either u is added to T before v or v before u. If u is added first, when processing u, the algorithm checks the edge (u, v) before backtracking. If v is unmarked at the time, (u, v) is added to T, and v is a child of u; if v is already marked visited at the time, it is marked between the start and end of the processing of u, and hence is a descendant of u.
The other case (when v is added first) follows the same argument. □