Learning Goals

- Definition of metrics
- Definition of Center Selection (a.k.a. k-center) Problem
- Understand the greedy algorithm
- Analyze the approximation ratio of the greedy algorithm

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- The problem is also known as the metric k-center problem.

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- $d(x,y) = ||x - p||_p = [\sum_j (x_j - y_j)^p]^{1/p}$, for $p \geq 1$, the ℓ_p distance.

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- Which sites should be considered "covered"?
- Suppose we are interested in whether it is possible to choose k centers with covering radius $\leq r$ for some r.
    - Alternatively, we may think of having guessed a covering radius r. Later we can look for an appropriate r by binary search.

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  - If we terminate with a non-empty R, declare failure; otherwise we find a set C, $|C| \leq k$, with a covering radius $\leq r$.
- Note that the algorithm is not fully "greedy": in each step s is chosen arbitrarily. It turns out that being more selective in that step does not help with the approximation ratio.

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Let C^* be any subset of S with covering radius $\leq \frac{r}{2}$, we show $|C^*| > k$.
Recall our algorithm terminated with a set of centers C, $|C| = k$, without covering all sites within distance r.

Proof (continued..)

For any site s and radius δ, let's denote by $B(s, \delta)$ the set of sites within distance δ to s, i.e., $B(s, \delta) := \{t \in S : d(s, t) \leq \delta\}$.

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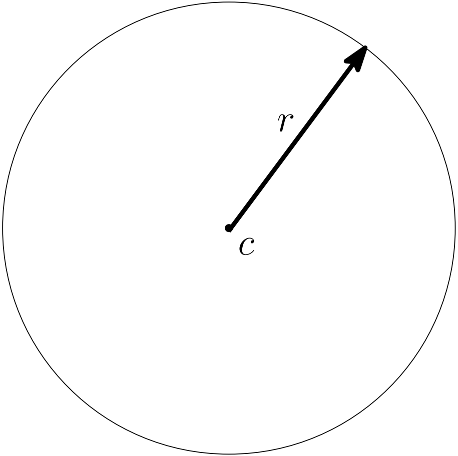
Key observation: $B(o_c, \frac{r}{2}) \subseteq B(c, r)$.

Proof by picture



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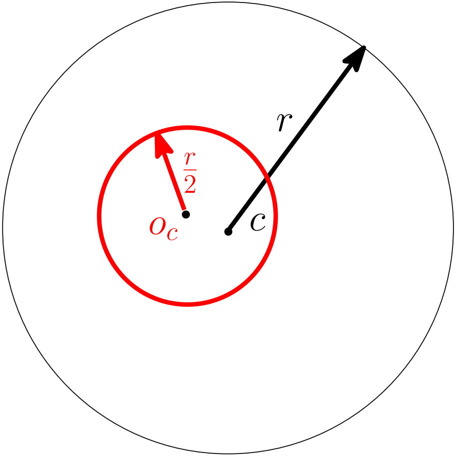

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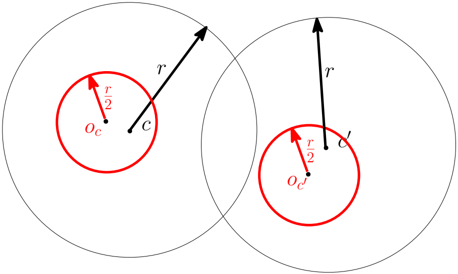

$c' \notin B(c, r)$, so $c' \notin B(o_c, \frac{r}{2})$, therefore $o_{c'} \neq o_c$.

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Also, $\cup_{c \in C} B(o_c, \frac{r}{2}) \subseteq \cup_{c \in C} B(c, r) \subsetneq S$;

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Also, $\cup_{c \in C} B(o_c, \frac{r}{2}) \subseteq \cup_{c \in C} B(c,r) \subsetneq S$;
But, by assumption, $\cup_{o \in C^*} B(o, \frac{r}{2}) = S$, so $|C^*| > |C| = k$. □

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Answer: It's NP-hard to get $(2 - \epsilon)$-approximation for any $\epsilon > 0$. (Think about the reduction from Vertex Cover.)