Learning Goals

- Define the class co-NP
- Understand the conceptual difference between NP and co-NP
- Show that NP $\neq$ co-NP $\Rightarrow$ P $\neq$ NP

Complement problems

Definition

For a decision problem A, its *complement* $\overline{A}$ is the following problem: an instance of A has answer YES if and only if the same instance has answer NO as an instance of $\overline{A}$.

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Example

$\overline{\text{SAT}}$: given a boolean formula in conjunctive normal form, the answer is YES if and only if the formula has no satsifying assignment.

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Alternatively, a problem is in co-NP if and only if there is a polynomial verifier and a polynomial $p(\cdot)$, such that for a NO instance, there exists a certificate of length $\leq p(\cdot)$ accepted by the verifier, and for a YES instance, no certificate of length $\leq p(\cdot)$ can make the verifier accept.

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- This is a stronger conjecture than P $\neq$ NP:

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- If $co\text{-}NP \subseteq NP$, there should be short certificates for boolean formulas having no satisfying assignment. Intuitively this doesn't sound likely.
- It is widely conjectured that $co\text{-}NP \neq NP$.
- This is a stronger conjecture than $P \neq NP$:

Proposition

$NP \neq co\text{-}NP \Rightarrow P \neq NP$.

Proof.

If $P = NP$, then one can solve problems in NP by solving their complement problems in NP, in polynomial time. □

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- Note that the distinction between NP and co-NP is possible only under Karp reduction and not under Turing reduction.
  - In other words, with Turing reduction, any problem in co-NP is easily reducible to its complement in NP, and vice versa.
- The problem $\overline{\text{SAT}}$ is a complete problem in co-NP. This is a direct consequence of Cook-Levin theorem.