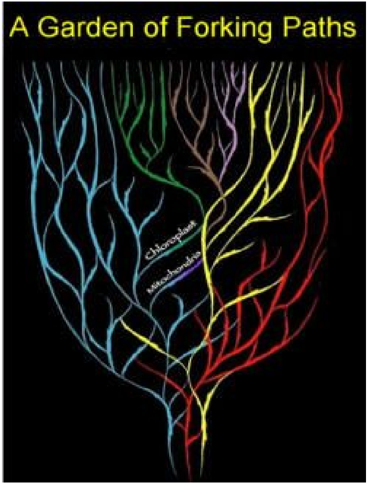
Learning Goals

- Basic definitions of finite probabilities: sample space, probability, events
- State and apply union bound.
- Define independence, and apply its properties in probability calculations
- Contention resolution with random access, and analysis of its efficiency

Borges's Garden of Forking Paths



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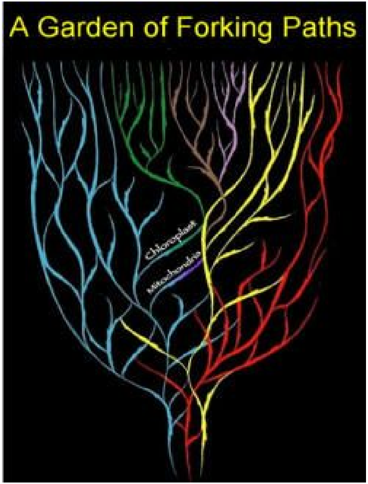

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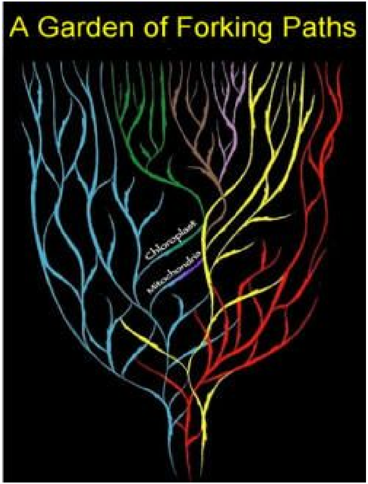

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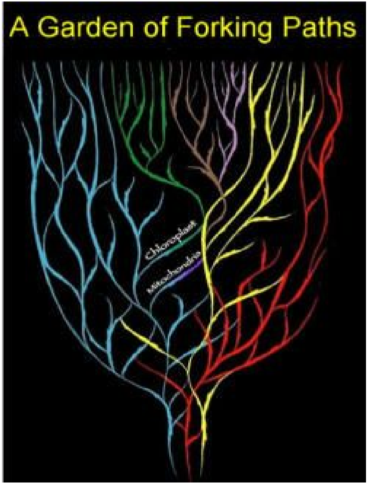

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- A probability space is defined by weights on those realizations.

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- If everything is fair, then each outcome has probability mass $1/36$.
- Let $\mathcal{E}$ be the event that the sum of the two numbers is 11, then $\mathcal{E} = \{(6,5),(5,6)\}$, so $\mathbf{Pr}[\mathcal{E}] = 1/18$.

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Exercise: If A and B are independent, then so are $\overline{A}$ and B, and so are $\overline{A}$ and $\overline{B}$.

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- We would like that all tasks to get served fast.
- Trivial if the tasks can agree on some ordering and requests the service one by one.
- Problem: The tasks cannot talk with each other and there is no central authority.
- **Randomized strategy:** In each time step, each task requests with some small probability p, *independently*.

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- Let $S[i, t]$ denote the event that task i sends a request at time t *and* gets served, then

$$\mathbf{Pr}\left[S[i, t]\right] = \mathbf{Pr}\left[A[i, t] \cap \bigcap_{j \neq i} \overline{A[j, t]}\right] = p(1 - p)^{n-1}.$$

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- To maximize $\mathbf{Pr}[S[i,t]]$, set $p = 1/n$.

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We set p to maximize $\mathbf{Pr}[S[i,t]]$ to $\frac{1}{n}(1-\frac{1}{n})^{n-1}$. How good is this?

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Proposition

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So $1/(en) \leq \mathbf{Pr}[S[i,t]] \leq 1/(2n)$. Therefore $\mathbf{Pr}[S[i,t]]$ is asymtotically $\Theta(1/n)$.

Waiting time for a particular task to succeed

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  - Remark: in many situations, the two give answers that are close: sometimes one may show that the random quantity *concentrates* around its expectation. *Tail bounds* are used to prove this.
- Probability with which task i does not succeed in the first t steps:

$$\mathbf{Pr}\left[\cap_{r=1}^{t}\overline{S[i,r]}\right] = \prod_{r=1}^{t}[1 - \mathbf{Pr}\left[S[i,r]\right]] = \left[1 - \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}\right]^{t}.$$

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- Big picture (useful rough estimations): if we have a biased coin that gives Heads with probability $1/k$:
  - In about k independent tosses, one "expects" to see a Heads;
  - However, with constant probability, a Heads doesn't show in k tosses;
  - But if one tosses the coin $\Theta(k\log k)$ times, the probability that no Heads shows up quickly tends to 0.

Waiting time for all tasks to succeed

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By the union bound, we have

$$\mathbf{Pr}\left[\cup_{i=1}^{n} F[i,t]\right] \leq \sum_{i=1}^{n} e^{-t/en} = ne^{-\frac{t}{en}}.$$

So for $t = \lceil 2en \ln n \rceil$, this is at most $\frac{1}{n}$.