Review of last lecture

- Bellman-Ford algorithm: a dynamic programming that
  1. Reports a negative cycle when one exists;
  2. Finds min-cost paths when no negative cycle exists.
- Running time: $O(mn)$: m operations per iteration, $O(n)$ iterations.
- Proof ideas:
  1. first i iterations find out min-cost paths with at most i edges
  2. Min-cost paths must be simple when there is no negative cycle

Learning Goals

- Definition of flow network and flows
- Motivation and definition of Residual graphs
- Ford-Fulkerson algorithm description
- Running time of Ford-Fulkerson

Flow Networks

A *flow network* is a directed graph $G = (V, E)$, which includes two special nodes: s, called the *source*, and t, called the *sink*. Each edge e is associated with a capacity $c_e > 0$.
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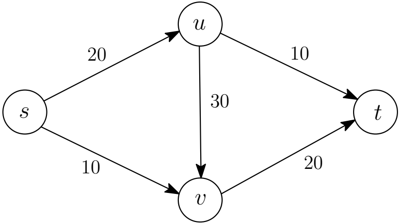

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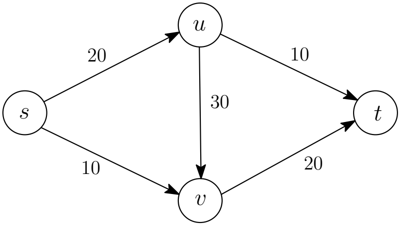

Notation: for $v \in V$, $\delta_{\mathrm{in}}(v)$ is the set of edges going into v, and $\delta_{\mathrm{out}}(v)$ the set of edges going out of v.

Flows

Definition

A *flow* is a function $f : E \to \mathbb{R}^+$ satisfying:

1. *Capacity conditions*: $\forall e \in E,\ 0 \leq f(e) \leq c_e$.
2. *Conservation conditions*: $\forall v \in V \setminus \{s, t\}$,

$$\sum_{e \in \delta_{\mathrm{in}}(v)} f(e) = \sum_{e \in \delta_{\mathrm{out}}(v)} f(e).$$

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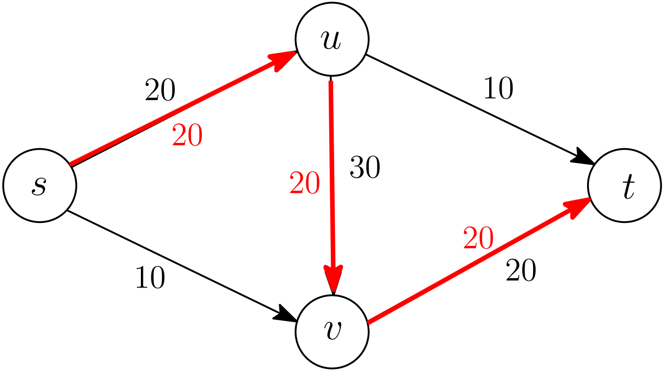
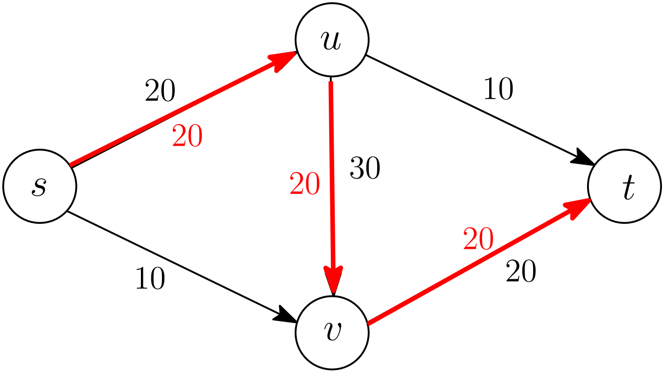
The maximum flow problem: given a flow network, compute a flow with maximum value.

Flow Example



Example of a flow (in red).

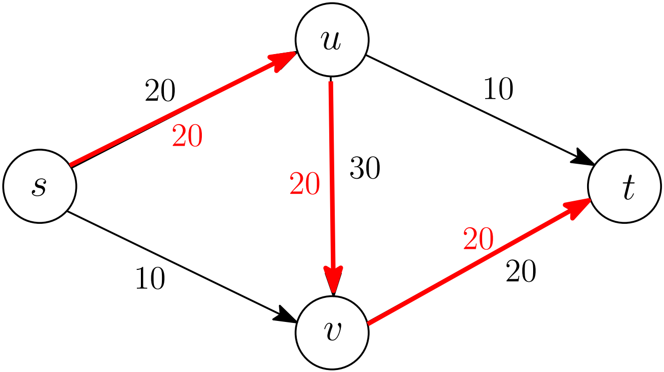
Flow Example



Example of a flow (in red).

The value of the flow is 20.

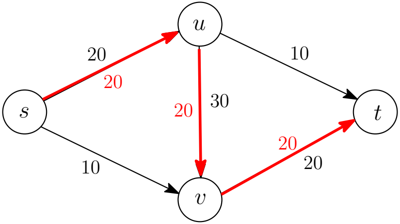
Flow Example



Example of a flow (in red).

The value of the flow is 20.
Is this a maximum flow?

Flow Example Cont.



Flow from previous page.

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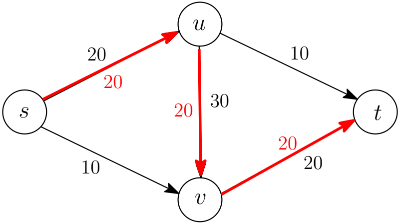

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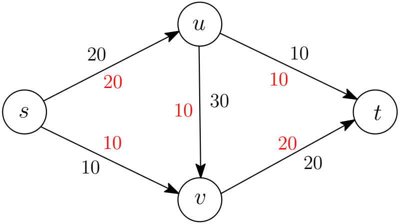
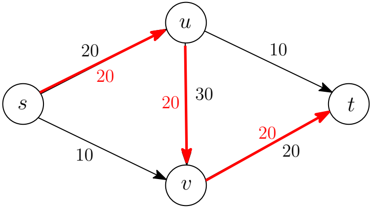


Maximum flow.

Residual Graph



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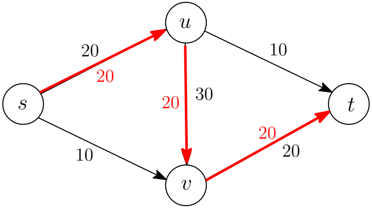

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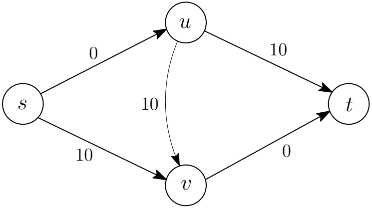

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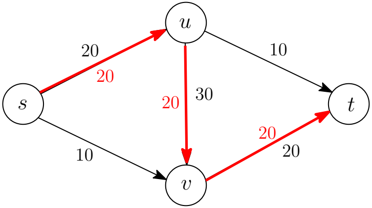
Residual Graph



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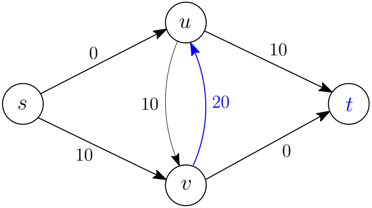

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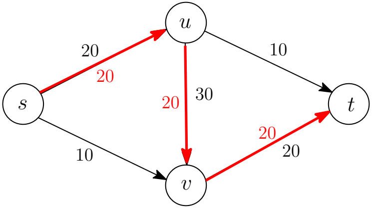

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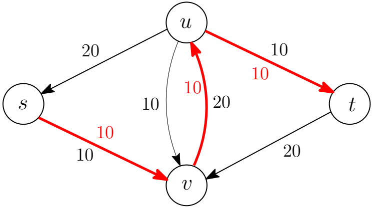

A flow pushed in the residual graph along a path.

The Residual Graph

Definition

Given a flow f in a flow network G, the *residual graph* G_f is defined as:

- G_f has the same set of nodes as G (including s and t);
- for each $e = (u, v)$ of G on which $f(e) < c_e$, e is in G_f and is called a *forward edge*; its capacity in G_f is $c_e - f(e)$;
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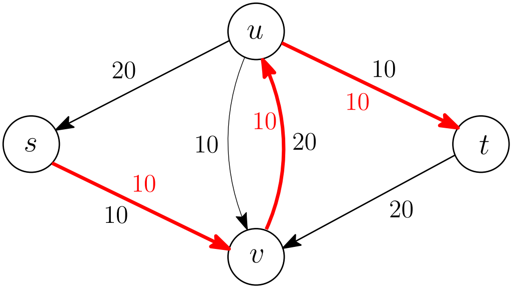
The capacities in G_f are called the *residual capacities*.
Note that the residual capacities are all strictly positive.

Augmenting paths

Definition

Any (simple) s-t path in the residual graph G_f is called an *augmenting path*. In an augmenting path P in the residual graph G_f, the minimum residual capacity is called the *bottleneck*, denoted as bottleneck(P, f).



(s, u, v, t) is an augmenting path; its bottleneck is 10.

Augmenting along a path

Given a flow f in G, if there is an s-t path P in its residual graph G_f, the following operations on f is called an *augmentation along P*

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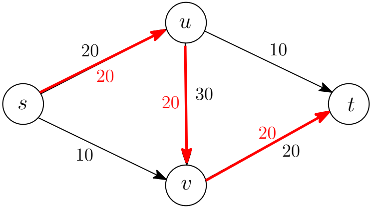
Proposition

The result of an augmentation, f', is a flow in G, with value $|f| + b$.

Augmentation example



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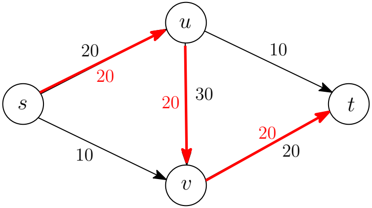

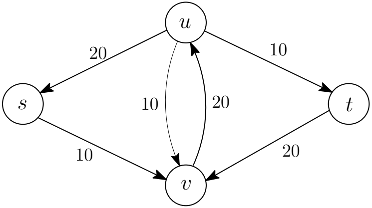

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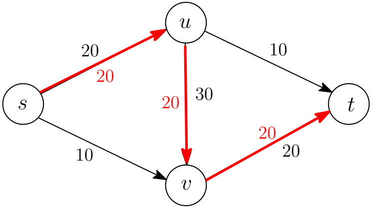

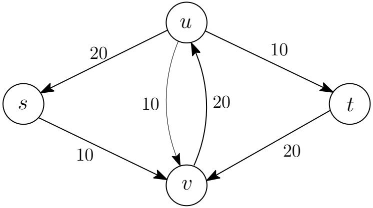

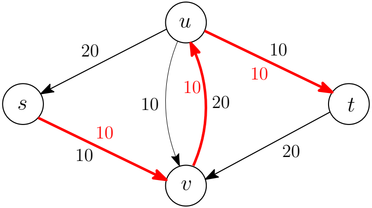

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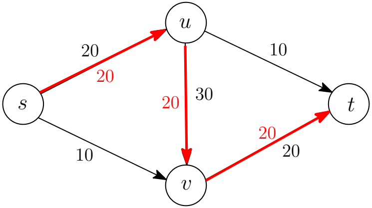

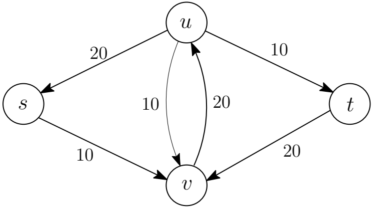

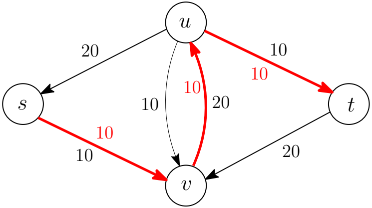

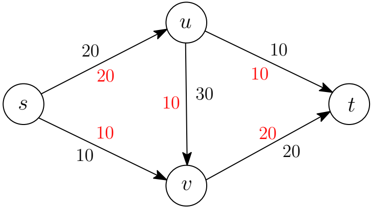

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An augmenting path starts from s, and $\delta_{\text{in}}(s) = \emptyset$, so the first edge in the path is forward.

$\square$

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