Learning Goals

- Define Hamiltonian Cycles and Paths
- State the HAMILTONIAN CYCLE/PATH problem
- Understand the reduction from HAMILTONIAN CYCLE to HAMILTONIAN PATH
- Understand the reduction from 3-SAT to HAMILTONIAN CYCLE
- State the TRAVELING SALESMAN problem
- Understand the reduction from HAMILTONIAN PATH to TRAVELING SALESMAN
- Definition of Eulerian paths and cycles
- Criterion for existence of Eulerian path/cycle

Classical NP-complete problems

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- Since then, new problems are shown to be NP complete almost every week (if not every day).
- Different classes of NP-complete problems enable us to recognize other NP-complete problems faster.
- Reductions to these problems are exemplary for coming up with reductions.

Hamiltonian Cycles

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In the HAMILTONIAN CYCLE problem, we are given a directed graph and must decide whether there exists a Hamiltonian cycle. In the HAMILTONIAN PATH problem, we are given a directed graph and must decide whether there exists a Hamiltonian path.

Theorem

HAMILTONIAN CYCLE is NP*-complete.*

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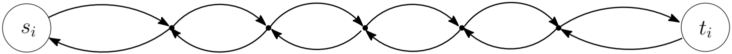
In HAMILTONIAN CYCLE, what is a gadget to represent the TRUE or FALSE assignment to a variable in 3-SAT?

Gadget representing variables



A gadget for a variable. Starting from s_i there is one way to traverse all nodes and arrive at t_i (TRUE); starting from t_i, there is one way to traverse all nodes and arrive at s_i (FALSE).

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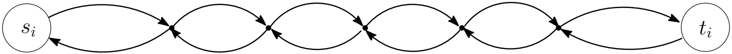

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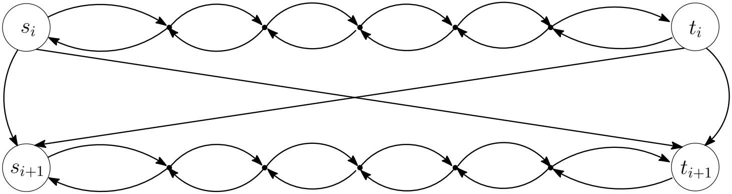
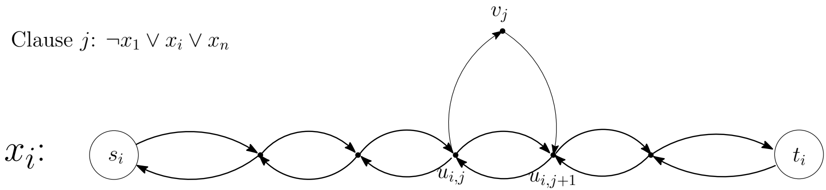


Connection of two variable gadgets.

Adding Clauses

Clause j: $\neg x_1 \vee x_i \vee x_n$



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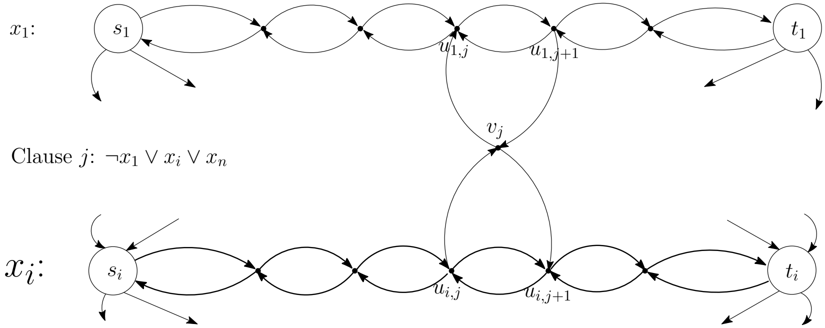

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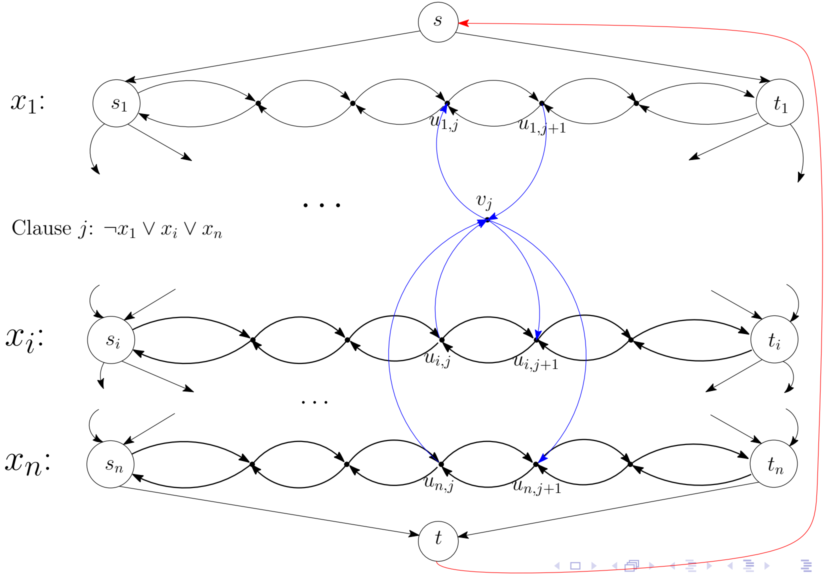
Adding starting and ending points



(Tentative) Formal Description

Given a 3-SAT formula, construct a directed graph G:

- Nodes of G:
  1. Add node $s = s_0$, $t = t_{n+1}$;
  2. For each variable x_i, add node $s_i = u_{i,0}, t_i = u_{i,m+1}, u_{i,1}, \ldots, u_{i,m}$;
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  5. Connect t to s.

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Claim

The given 3-SAT formula has a satisfying truth assignment if and only if G has a Hamiltonian cycle.

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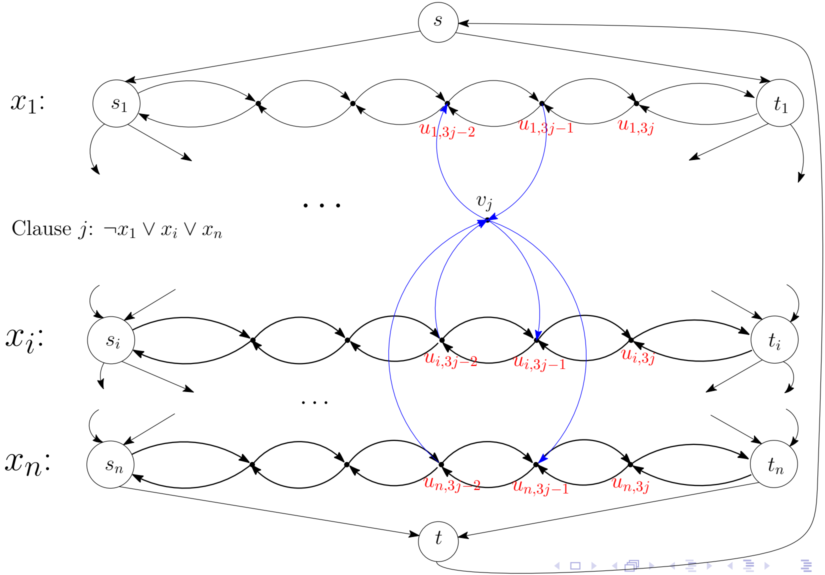
Idea: Refine the construction to prevent such jumping from happening.

Refined Construction



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  3. For each clause j that includes literal x_i, connect $u_{i,3j-2}$ to v_j and v_j to $u_{i,3j-1}$;
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The way each x_i cycle is traversed by the Hamiltonian cycle now corresponds to a truth assignment. It is straightforward to verify that all clauses are satisfied by this assignment. □

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Proof Sketch.

Pick one node and split it into two copies, one with all incoming edges and the other with all outgoing edges. □

Eulerian Tours and Eulerian Circuits

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Theorem

Deciding whether a given graph has a Eulerian cycle is in P*.*
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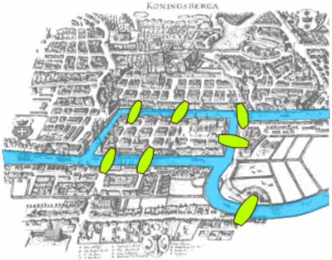
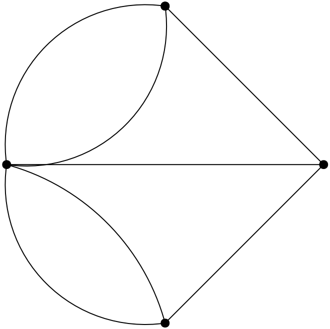
Seven Bridges of Königsberg





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Proof sketch for a graph with two odd-degree nodes: First find a path from one odd degree node to the other, then the rest of the graph has a Eulerian cycle in each of its connected component. Concatenate these cycles to the path forms a Eulerian path.

Traveling Salesman Problem

Definition

Given n nodes $v_1, \ldots, v_n$, a *tour* is a path that starts from v_1, visits every other node exactly once, and returns to v_1.
In the *Traveling Salesman Problem (TSP)*, we are given n nodes and a distance $d_{i,j}$ from each node v_i to another one v_j, and a bound D. We are asked to decide whether there is a tour of total distance at most D.

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Last remarks on TSP

Remark: When $d_{i,j} = d_{j,i}$, $\forall i, j$, the problem is called a *symmetric TSP* problem; otherwise it is said to be asymmetric.

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We showed that asymmetric, metric TSP is NP-complete.
In fact, symmetric, metric TSP is already NP-complete.

Exercises

Does the following problem admit a polynomial-time algorithm or is it NP-complete?

1. Given a set $A = \{a_1, \dots, a_n\}$, a collection $B_1, B_2, \cdots, B_m$ of subsets of A, and an integer $k > 0$. Is there a set $H \subseteq A$, $|H| \leq k$ such that $H \cap B_i \neq \emptyset$ for $i = 1, \dots, m$?

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   2. Given m paths $P_1, \cdots, P_m$ from s to t, does there exist at least k paths among $P_1, \cdots, P_m$ that are edge-disjoint?

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