Learning Goal

- Massaging mixed-signed objectives
- Recognizing disguised min cut problems
- Constructing flow networks with desired cut capacities

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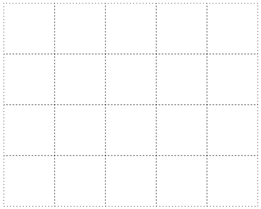
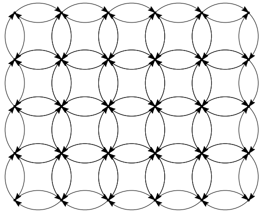
- Task: partition the pixels into foreground and background, to maximize the sum of likelihood scores and minimize the penalty for separating neighbors

Problem formulation

- Input: undirected graph $G = (V, E)$, with the nodes V representing the pixels, and an edge $\{i, j\}$ exists if pixels i and j are adjacent.
- Output: A partition of V into two subsets A and B, to maximize

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E, |A \cap \{i,j\}| = 1} p_{ij}.$$





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Add $\sum_{i \in V}(a_i + b_i)$, which is a constant!

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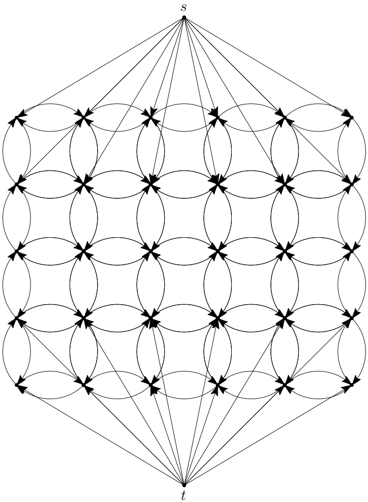
- Construct flow network G', with node set $V \cup \{s, t\}$.
- Add an edge from s to each node $i \in V$ with capacity a_i, an edge to t from each node i with capacity b_i.
- For each edge $\{i, j\}$ in G, add edges (i, j) and (j, i) in G', with capacity p_{ij}.

Example flow network



Proof of Reduction

Claim

For any s-t cut (A', B') in G',
$c(A', B') = \sum_{i \in A' \setminus \{s\}} b_i + \sum_{j \in B \setminus \{t\}} a_j + \sum_{\{i,j\} \in E : |A' \cap \{i,j\}| = 1} p_{ij}$.