Learning Goals

- A simple randomized algorithm computing min cuts
- Las Vegas and Monte Carlo algorithms
- Conditional probabilities and their use in the analysis of randomized algorithms
- Improving the precision of a randomized algorithm by repetition

The Min Cut Problem

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- Given an undirected graph $G = (V, E)$, for $S \subsetneq V$, $S \neq \emptyset$, let $c(S)$ be $|\{(u, v) \in E : u \in S, v \notin S\}|$, called the *volume* of the *cut* $(S, \overline{S})$.

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- Question: find S that minimizes $c(S)$.
- Deterministic polynomial-time algorithm: fix $s \in V$, then for each $t \neq s$, find the minimum number of edge-disjoint paths from s to t, corresponding to an "s-t" cut with a volume equal to the number of such paths. Take the one with the smallest volume.

A simple randomized algorithm due to David Karger

- Choose an edge (u, v) uniformly at random and *contract* it: merge u and v into a new node w, which inherits all other edges incident to u or v.
    - The resulting graph may have parallel edges: if $(s, u), (s, v) \in E$, then there are two parallel edges between s and w in the new graph.

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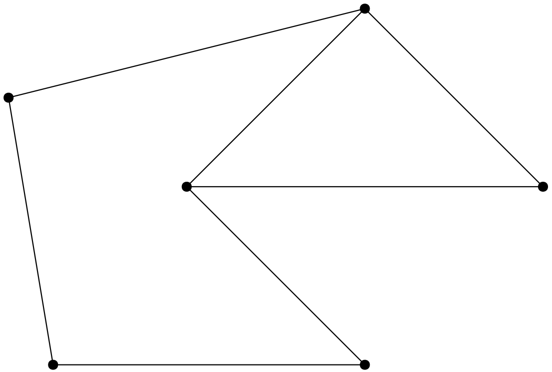
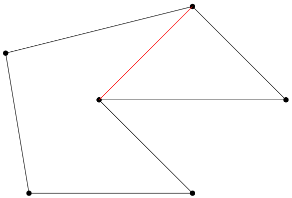
- Repeat this until there are two only nodes, corresponding to a partition of V.

Example



Input graph

Example



Randomly select an edge

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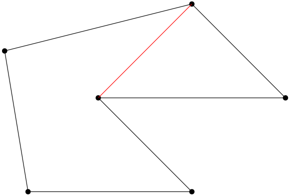

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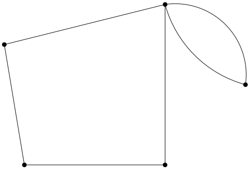

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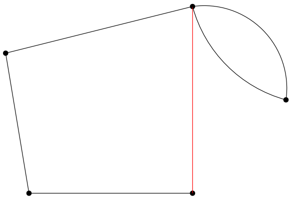

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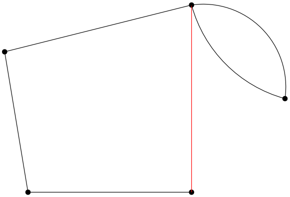

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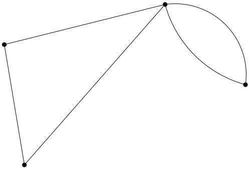

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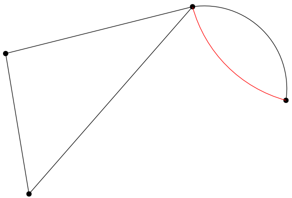

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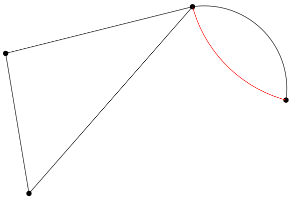

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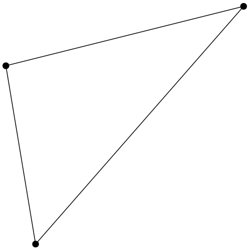

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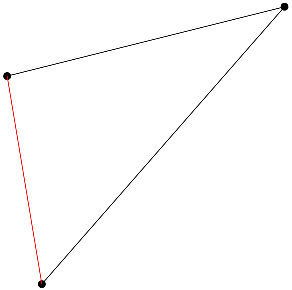

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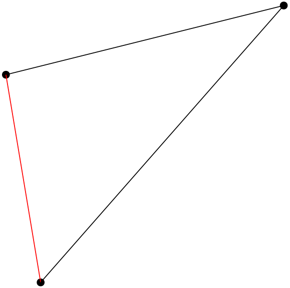

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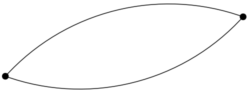

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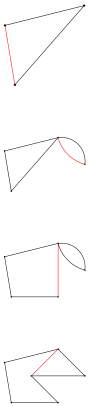

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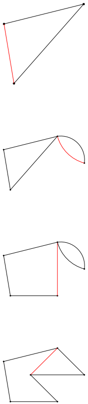

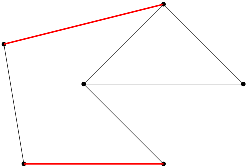

Cut output

Two types of randomized algorithms

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- A randomized algorithm is *Monte Carlo* if it may return incorrect solutions sometimes.
- The min cut algorithm we just saw is Monte Carlo.

Theorem

The algorithm finds a minimum cut with probability at least $1/\binom{n}{2}$, *where* $n = |V|$.

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- Repeat many times independently. The probability of being incorrect in all these trials diminishes exponentially.
- If an algorithm succeeds with probability p, with after t independent trials, the probability we never see a correct solution is $(1-p)^t$.
- For the min cut algorithm, let t be $\Omega(n^2 \log n)$, then the probability of seeing t incorrect solutions in a row is at most

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^{\Omega(n^2 \log n)} = e^{-\Omega(\log n)} = n^{-\Omega(1)}.$$

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- Let $\mathcal{E}_i$ be the event that, in the i-th step, an edge not in C is chosen. Then we need to show $\mathbf{Pr}[\cap_{i=1}^{n-2}\mathcal{E}_i] \geq 1/\binom{n}{2}$.

Brief review of conditional probabilities

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- Example 4: Total probability rule:

$$\mathbf{Pr}\left[A\right] = \mathbf{Pr}\left[A \mid B\right]\mathbf{Pr}\left[B\right] + \mathbf{Pr}\left[A \mid \overline{B}\right]\mathbf{Pr}\left[\overline{B}\right].$$

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  - Answer: $1/3$. A less deceptive version: Conditioning on that not both of them are girls, what is the probabilty that both are boys?

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  - Answer: 1/3. A less deceptive version: Conditioning on that not both of them are girls, what is the probabilty that both are boys?
  - Let the atom events be BB, BG, GB, GG, each with probability 1/4. Then event A, "one of them is a boy", is {BB, BG, GB}, and event B, "both are boys", is {BB}. So $\mathbf{Pr}[B \mid A] = \frac{|A \cap B|}{|A|} = 1/3$.

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- Therefore,

$$\begin{aligned}
\mathbf{Pr}\left[\cap_{i=1}^{n-2}\mathcal{E}_i\right] &= \mathbf{Pr}\left[\mathcal{E}_1\right] \cdot \mathbf{Pr}\left[\mathcal{E}_2 \mid \mathcal{E}_1\right] \cdots \mathbf{Pr}\left[\mathcal{E}_{n-2} \mid \cap_{i=1}^{n-3}\mathcal{E}_i\right] \\
&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\cdots\left(1 - \frac{2}{3}\right) \\
&= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{2}{4} \cdot \frac{1}{3} \\
&= \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}.
\end{aligned}$$