Learning Goals

- Definition of Fully Polynomial-Time Approximation Schemes (FPTAS)
- Design pseudo-polynomial time dynamic programming algorithms for NP-hard problems
- Apply rounding to DP and analyze it to obtain approximation algorithms

The Knapsack Problem

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- We already showed the decision version to be NP-complete.

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- Exercise: Modify the greedy algorithm and get a 2-approximation with a greedy approach (Question 3 in PS6 is a special case)

Attempt at Approximation: Dynamic Programming (DP)

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- Approximation idea: Can we round the weights, by replacing the actual weights by multiple of an appropriate integer b?
- Example: If the weights are 5, 24, 77, 131, 142, with $W = 156$, round weights to 0, 25, 75, 125, 150, and $W = 150$?

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- Approximation idea: Can we round the weights, by replacing the actual weights by multiple of an appropriate integer b?
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- Problem: weights are hard constraints; by rounding them, easily lead us to infeasible solutions (if we round weights down) or bad approximations (if we round weights up).
- Alternative: Such problems won't arise if we round values instead. We need a new DP that has a pseudopolynomial dependence on values instead of weights.

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Running time: for each item i, we go through the array of length $O(iv^*)$, so total running time $O(v^* n^2)$.

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- How good an approximation is S, the set of items chosen by the algorithm?
- Let S^* be any other feasible set of items (think it as the optimal solution), then

$$\sum_{i \in S^*} v_i \leq \sum_{i \in S^*} \tilde{v}_i = b \sum_{i \in S^*} \hat{v}_i \leq b \sum_{i \in S} \hat{v}_i = \sum_{i \in S} \tilde{v}_i \leq nb + \sum_{i \in S} v_i.$$

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- Running time: $O(n^2 v^*/b) = O(n^3 \epsilon^{-1})$.

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For any $\epsilon > 0$*, the Knapsack problem can be approximated to a factor of* $1 + \epsilon$ *by an algorithm that runs in time* $O(n^3\epsilon^{-1})$.

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A family of approximation algorithms is a *polynomial-time approximation scheme* (PTAS) for an optimization problem if for any $\epsilon > 0$, there is an algorithm in the family that is a $(1 + \epsilon)$-approximation algorithm for the problem, with polynomial running time when ϵ is treated as a constant. If the running time depends polynomially on ϵ^{-1}, the family is said to be a *fully polynomial-time approximation scheme* (FPTAS).

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We have obtained an FPTAS for the Knapsack problem.