Theorem (Max-Flow Min-Cut Theorem)

The following statements are equivalent:

1. *f is a maximum flow on a flow network G;*
2. *There is an s-t cut (A, B) with $c(A, B) = |f|$;*
3. *There exists no augmenting path in the residual graph G_f.*

Proof.

$2 \Rightarrow 1$: cut capacities are upper bounds for flow values.
$1 \Rightarrow 3$: Augmenting along a path increases a flow's value.
$3 \Rightarrow 2$: The set of nodes reachable from the source in G_f gives the cut. □

Learning Goals

- Matching definition
- Reduction from bipartite matching to max flow
- Hall's theorem and its proof

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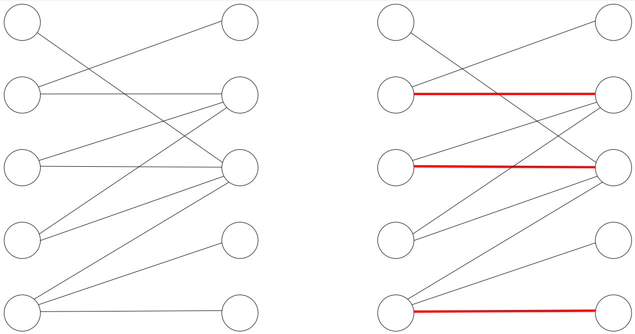
Definition

Given an undirected graph $G = (V, E)$, a set of edges $M \subseteq E$ is a *matching* if each node in V is incident to at most one edge in M.

Maximum Bipartite Matching Problem

Problem (The unweighted maximum bipartite matching problem)

- *Input: a bipartite graph $G = (U, V, E)$. (Recall: this means all edges have one endpoint in U and the other in V.)*
- *Output: a matching M with the maximum cardinality.*



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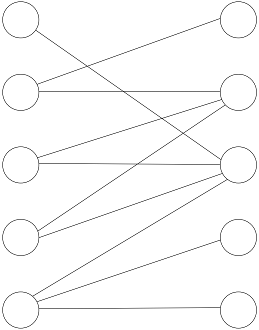

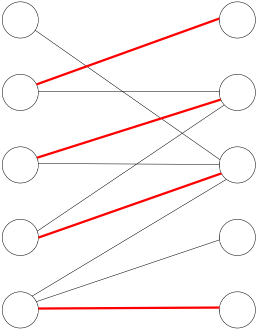

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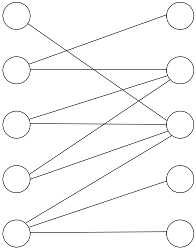
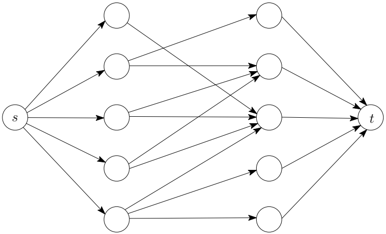
- For every $(u, v) \in E, u \in U, v \in V$, add directed edge (u, v) to G', with capacity 1.





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3. Special case of 2.

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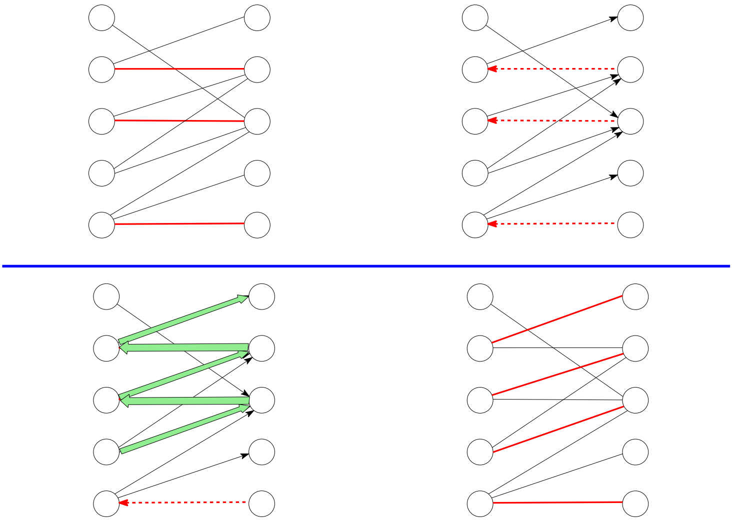
Illustration of a step from the algorithm



Hall's Theorem

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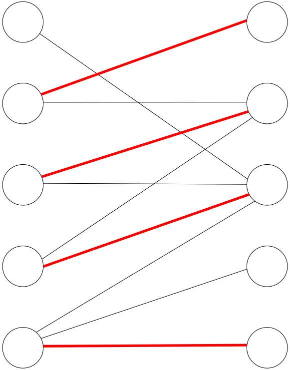

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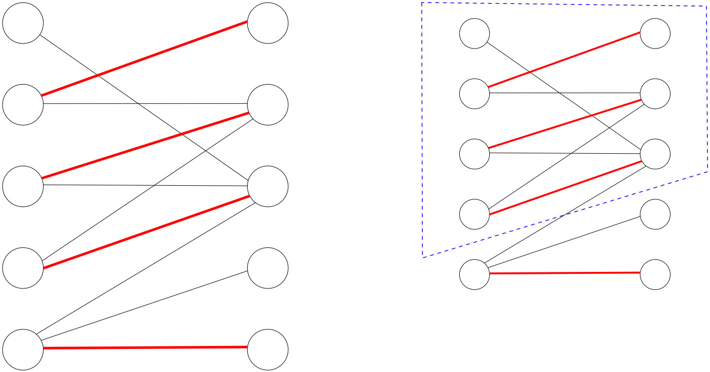

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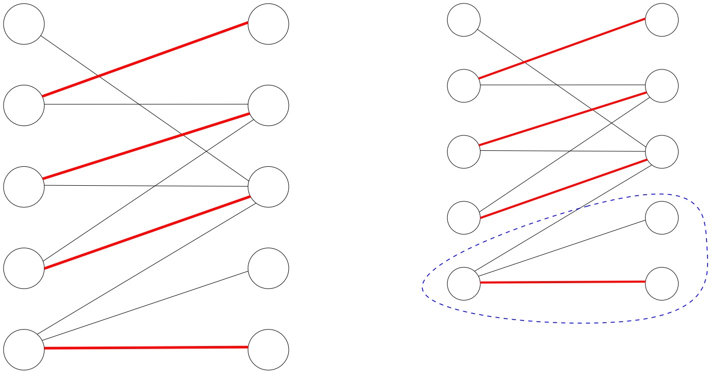

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$$|U| > c(A, B) = |U \setminus A| + |A \cap V| \geq |U \setminus A| + |\Gamma(A \cap U)|.$$
$$\Rightarrow |U| - |U \setminus A| = |A \cap U| > |\Gamma(A \cap U)|.$$

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- Similar ideas (of augmenting along a collection of shortest paths that "block" s from t) lead to faster algorithms for the max flow problem: Dinic's algorithm, running in time $O(mn^2)$.
- (The algorithm by Edmonds and Karp that run in time $O(m^2n)$ is an important predecessor.)