Learning Goals

- Definition of cuts and their capacities
- Cut capacities are upper bounds on flow values
- Max Flow Min Cut Theorem and its proof
- Correctness of Ford-Fulkerson
- Ford-Fulkerson as an algorithm to find min cuts
- Properties of Ford-Fulkerson

Proof of Correctness for Ford-Fulkerson

We already know:

- After each augmentation, we have a new flow with more value.
- For integral capacities, the algorithm terminates after at most C rounds, where $C = \sum_{e \in \delta_{\text{out}}(s)} c_e$.

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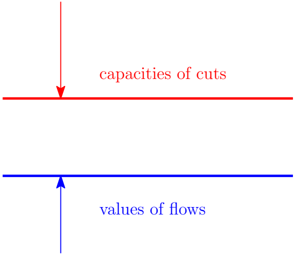

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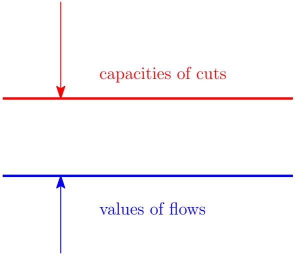

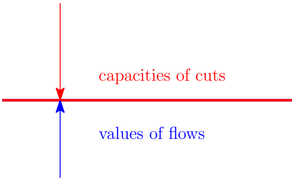

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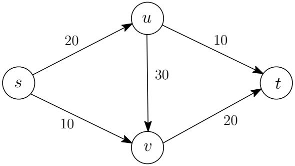

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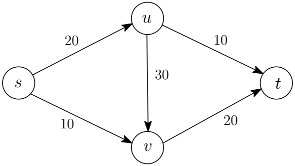

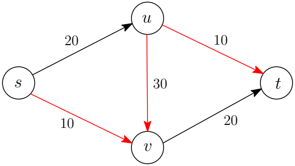

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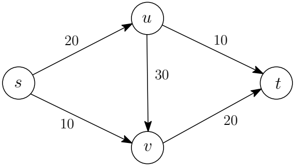

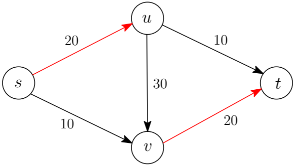

$c(\{s, v\}, \{u, t\}) = 40.$

Flow across cuts

Given a flow f and an s-t cut (A, B), define

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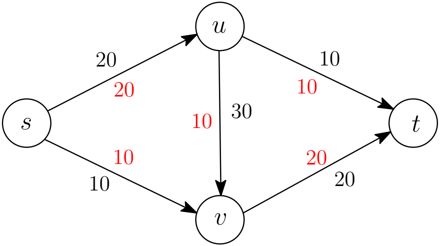

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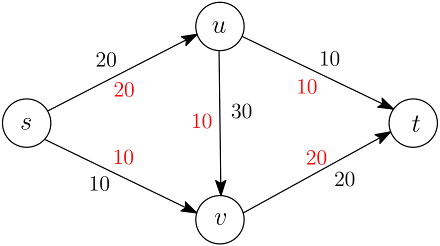

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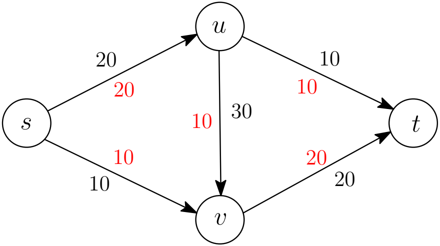

$$f^{\text{out}}(\{s, v\}) = 40;$$
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$$f^{\text{out}}(\{s, u\}) = 30;$$
$$f^{\text{in}}(\{s, u\}) = 0.$$

Relating Flows to Cuts

Lemma

For any s-t cut (A, B) and any flow f, the value of f, $|f|$, is $f^{out}(A) - f^{in}(A)$.

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For any s-t cut (A, B) *and any flow* f*,* $|f| \leq c(A, B)$*. If* $|f| = c(A, B)$*, then* $f^{\mathrm{in}}(A) = 0$*, and* $f(e) = c_e$ *for each* $e \in \delta_{\mathrm{out}}(A)$.

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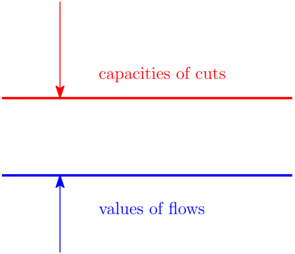

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$$|f| = f^{\text{out}}(A) - f^{\text{in}}(A) \leq f^{\text{out}}(A) = \sum_{e \in \delta_{\text{out}}(A)} f(e) \leq \sum_{e \in \delta_{\text{out}}(A)} c_e = c(A, B).$$

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If $|f| = c(A, B)$, all inequalities above must be tight. □

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By induction on $|A|$. IH: Lemma statement. Base case: $|A| = 1$, $A = \{s\}$, $f^{\text{in}}(A) = 0$, $f^{\text{out}}(A) = \sum_{e \in \delta_{\text{out}}(s)} f(e) = |f|$ by definition.

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$$f^{\text{out}}(A) - f^{\text{in}}(A) = f^{\text{out}}(A \setminus \{v\}) - f^{\text{in}}(A \setminus \{v\}) + \left(\sum_{e \in \delta_{\text{out}}(v)} f(e) - \sum_{e \in \delta_{\text{in}}(v)} f(e) \right) = |f|.$$

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Theorem (Max-Flow Min-Cut)

The following statements are equivalent:

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Corollary

When the Ford-Fulkerson algorithm terminates, the flow it returns is a maximum flow.

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$3 \Rightarrow 2$: Let S be the set of nodes reachable from s in G_f.
Claim: $c(S, \overline{S}) = |f|$.

Proof of Theorem ($3 \Rightarrow 2$)

Claim (Restatement)

If for a flow f, there is no augmenting path in the residual graph G_f, let S be the set of nodes reachable from s in G_f, then $c(S, \overline{S}) = |f|$.

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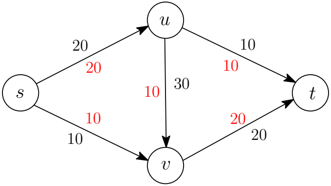
Therefore $f^{\text{out}}(S) = \sum_{e \in \delta_{\text{out}}(S)} c_e = c(S, \overline{S})$, $f^{\text{in}}(S) = 0$, and $|f| = f^{\text{out}}(S) - f^{\text{in}}(S) = c(S, \overline{S})$.

□

Illustration of 3 $\Rightarrow$ 2



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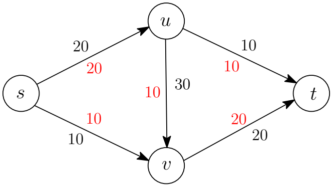

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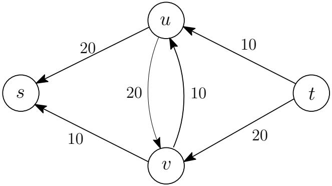

Residual graph

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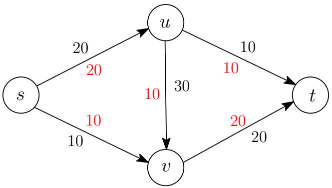

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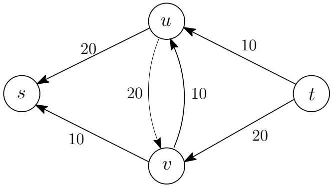

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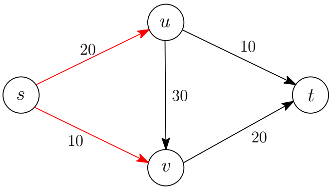

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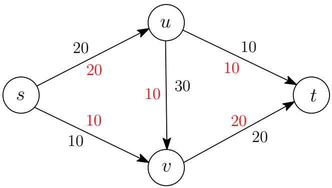

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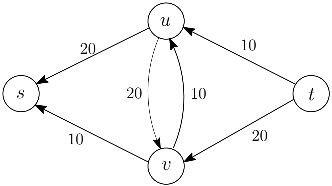
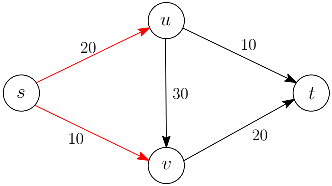
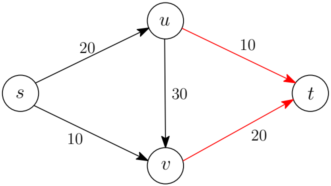


Residual graph



$S = \{s\}$.



Another min cut.

The Min Cut Problem

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- *Input: A flow network* $G = (V, E)$.
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- Such a cut is called a *min cut*.

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- Given a flow, verifying whether it is a max flow takes only $O(m)$ time.