Learning Goals

- Define class P and NP.
- Understand the relationship between P and NP.
- Define what is an NP-complete problem.
- State the decision problems SAT and 3-SAT.
- State Cook-Levin Theorem.
- Master the procedure to prove a problem is NP-complete.
- Understand the reduction from 3-SAT to INDEPENDENT SET.

The classes P

Note: In this lecture we cannot get into the nuts and bolts of some definitions or theorems, because we haven't defined a computation model (e.g. Turing machine). Nevertheless, everything stated can be rigorously proved.

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Recall from 221/320: by polynomial time, we mean polynomial in the length of the input.

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Recall from 221/320: by polynomial time, we mean polynomial in the length of the input.

Example: decision versions of Shortest Path, Minimum Spanning Tree, Max Flow, Min Cut, Bipartite Matching, Baseball Elimination...

The class NP

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1. if the answer to an instance a of A is YES, then there exists a *polynomial-length certificate* $c(a)$, such that V, when provided with both the instance a and the certificate, will return yes;
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Example: INDEPENDENT SET

- Input: graph G, integer k
- Certificate: a set S of nodes in G
- Verifier: check whether S is an independent set, and whether $|S| \geq k$. If so, return YES; if not, return NO.

Relationship between P and NP

Proposition

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Proof.

Given any problem in P, let the verifier V be a polynomial-time algorithm that solves the problem . Let the certificate be $\emptyset$. $\square$

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One of the most famous questions in (theoretical) computer science.

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Some philosophical discussion.

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Formally, a problem A is NP-complete if $A \in \mathrm{NP}$ and, $\forall B \in \mathrm{NP}$, $B \leq_{\mathrm{P}} A$.

SAT and Cook-Levin Theorem

Definition

In a *Boolean satisfiability (SAT)* problem, we are given a Boolean formula in *conjunctive normal form (CNF)*; that is, the formula is the AND of (many) OR clauses. We must decide whether there is a way of assigning TRUE and FALSE to each variable so that the formula evaluates to TRUE.

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A meaningful proof needs a rigorous definition of Turing machines.

Review of last lecture

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Proof sketch: If the polynomial-time reduction from A to B runs in time $p_1(\cdot)$, and the reduction from B to C runs in time $p_2(\cdot)$, then A can be solved by concatenating the reductions, with oracle access to C, and running time $O(p_1(\cdot)p_2(\cdot))$, which is still polynomial.

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We need to show that, for any $C \in \mathrm{NP}$, $C \leq_{\mathrm{P}} A$.

- Since B is NP complete, $C \leq_{\mathrm{P}} B$;
- But $B \leq_{\mathrm{P}} A$, therefore $C \leq_{\mathrm{P}} A$ by proposition.

$\square$

Procedure to show NP completeness

Given a problem A, to show it is NP-complete, we show that

1. A is in NP. We show a polynomial-time verifier: for TRUE instances, show polynomial-length certificates that makes the verifier accept, and for FALSE instances, show the verifier never accepts;
2. Take an NP-complete problem B, and show $B \leq_{\mathrm{P}} A$. To do this, we
   - Give a *polynomial-time* algorithm φ which takes as input an instance of B and outputs an instance of A;
   - Show that an instance b of B has answer TRUE if and only if the instance $\varphi(b)$ has answer TRUE.

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    1. For each clause, construct three nodes representing the three literals;
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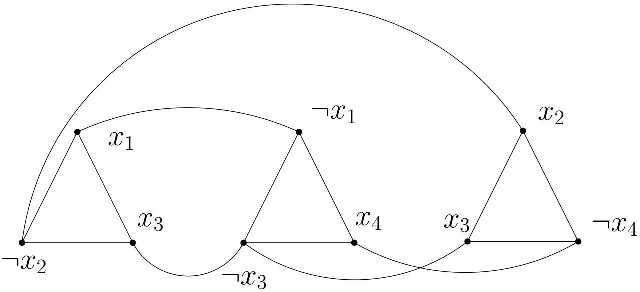
    3. For each clause, connect the three nodes representing its literals.

   Now show that the 3-SAT formula is satisfiable if and only if G has an independent set of size at least m.

Example instance



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Proof cont.

Claim

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3-SAT satisfiable $\Rightarrow$ G having independent set S of size m:

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S is an independent set:

1. No edge in any triangle is in $E(S)$;
2. No edge connecting a variable and its negate is in $E(S)$.

Proof cont.

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Proof.

G having an independent set S of size $m \Rightarrow$ 3-SAT formula satisfiable:
S must have one node from each 3-cycle corresponding to a clause.
Construct a truth assignment by letting the corresponding literal be TRUE. (After this, if some variables don't have an assignment, give them arbitrary assignment.)

1. There is no contradiction in this assignment.
2. All clauses are satisfied.

A little summary

- We have shown: 3-SAT $\leq_P$ INDEPENDENT SET $\leq_P$ VERTEX COVER $\leq_P$ SET COVER
- These problems are all clearly in NP.
- Both SAT and 3-SAT are NP complete
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- Therefore all these problems are NP complete.
- Note that VERTEX COVER can be solved in polynomial time for bipartite graphs.
- For non-bipartite graphs, maximum matching can still be solved in polynomial time. But the size of the smallest vertex cover can be strictly larger than the size of the maximum matching.