# Learning Goals

- State the 3-DIMENSIONAL MATCHING problem
- Understand the reduction from 3-SAT to 3-DIMENSIONAL MATCHING
- Define graph coloring and the chromatic number of a graph
- State the 3-COLORING problem

# 3-Dimensional Matching

## Definition

Given disjoint sets $X$, $Y$ and $Z$, each of size $n$, and given a set $T \subseteq X \times Y \times Z$ of ordered triples, a *3-dimensional matching* is a set $M$ of $n$ triples in $T$ so that each element of $X \cup Y \cup Z$ is contained in exactly one triple in $M$.

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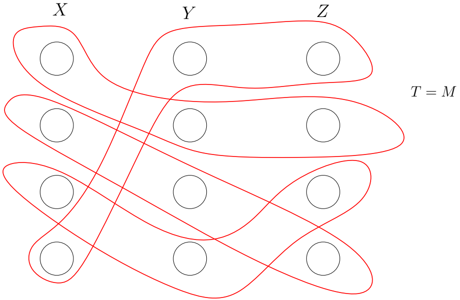

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3-DIMENSIONAL MATCHING $\in$ NP:

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We will show 3-SAT $\leq_P$ 3-DIMENSIONAL MATCHING.

# Proof Continued

**Proposition**

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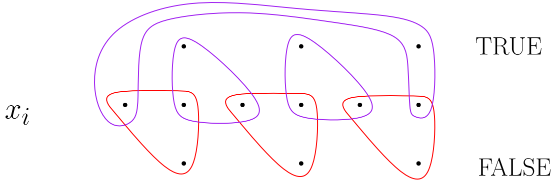
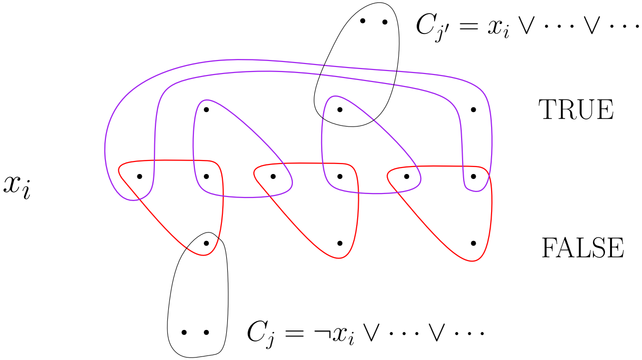
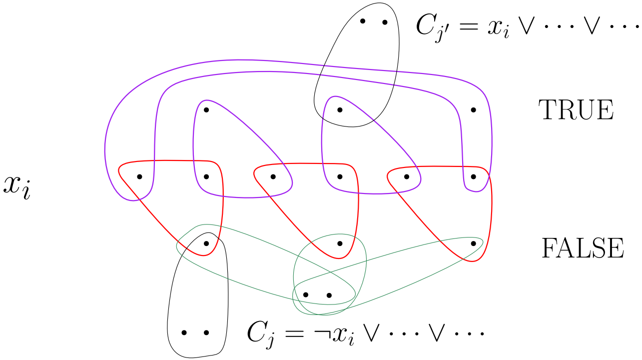


# Incorporating Clauses




Add two nodes for each clause $j$. If $x_i$ is in the clause, add a triple including the two clause nodes and the $j^{\text{th}}$ "TRUE node" of $x_i$; if $\neg x_i$ is in the clause, add a triple including the two clause nodes and the $j^{\text{th}}$ "FALSE node" of $x_i$.

# Adding cleanup nodes




Each clean up gadget consists of two cleanup nodes; add a triple for each TRUE or FALSE node of each $x_i$. How many of these are needed? To be calculated.

# Putting things together

Given a 3-SAT formula with $m$ clauses, we construct a 3-DIMENSIONAL MATCHING instance:

- Node set: (we first describe all the nodes and partition them into $X, Y, Z$ later)
  - For each variable $x_i$, create nodes $a_{i1}, \ldots, a_{i,2m}$, $T_{i,1}, \cdots, T_{i,m}$ and $F_{i,1}, \cdots, F_{i,m}$;
  - For each clause $j$, create nodes $c_{j1}$ and $c_{j2}$;
  - Create a number of cleanup gadgets; the $k^{\text{th}}$ cleanup gadget has two nodes $q_{k1}$ and $q_{k2}$.

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  - Create a number of cleanup gadgets; the $k^{\text{th}}$ cleanup gadget has two nodes $q_{k1}$ and $q_{k2}$.
- Triple set $T$:
  - For each $x_i$, for $j = 1, \cdots, m$, create triples $(a_{i,2j-1}, a_{i,2j}, F_{i,j})$ and $(a_{i,2j}, a_{i,2j+1}, T_{i,j})$ (interpreting $a_{i,2m+1}$ as $a_{i,1}$)
  - For each clause $j$, for each positive literal $x_i$ in it, add triple $(c_{j1}, c_{j2}, T_{ij})$; for each negative literal $\neg x_i$ in the clause, add triple $(c_{j1}, c_{j2}, F_{ij})$;
  - For each cleanup gadget $k$, for each $x_i$ and clause $j$, add $(q_{k1}, q_{k2}, T_{ij})$ and $(q_{k1}, q_{k2}, F_{ij})$.

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- Partition the nodes into $X, Y, Z$.
  - All the odd numbered "$a, c, q$" nodes form $X$;
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**Claim**

The 3-SAT formula is satisfiable if and only if the instance we constructed has a 3-dimensional matching.

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**Proposition**

$\chi(G) \leq 2$ if and only if $G$ is bipartite, and this can be checked in linear time.

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**Theorem**

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For a (simple) graph $G = (V, E)$, its *complement graph* $\overline{G}$ is has the same node set $V$, and any two distinct vertices $u$ and $v$ are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

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**Proposition (Exercise)**

$\chi(G) \geq \alpha(\overline{G})$.

# Exercises on coloring

**Proposition (Exercise)**

$\chi(G) \geq \alpha(\overline{G})$.

**Proof.**

An independent set of size $k$ in $\overline{G}$ is a *clique* of size $k$ in $G$, i.e., every pair of these $k$ nodes are connected. $k$ colors are needed for them. □

**Proposition (Exercise)**

3-COLORING $\leq_P$ $k$-COLORING

**Proof.**

Given a graph $G$, add $k-3$ nodes to $G$, all connected with each other and all connected with each node in $G$. The new graph is $k$ colorable if and only if $G$ is 3-colorable. □

# Exercises

**Question**

Give a graph $G$ where $\chi(G) > \alpha(\overline{G})$.

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In the 3-EXACT COVER problem, we are given a universe $U$ and a list of subsets $S_1, \cdots, S_m \subseteq U$ each of cardinality 3. We must decide whether there exists a subset of these subsets that cover each element in $U$ exactly once. Show that 3-EXACT COVER is NP complete.

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    2. Given $m$ paths $P_1, \cdots, P_m$ from $s$ to $t$, does there exist at least $k$ paths among $P_1, \cdots, P_m$ that are edge-disjoint?

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