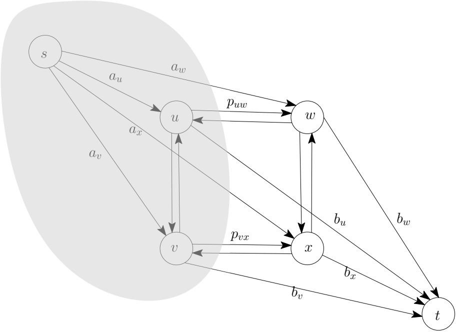
Review of Last Lecture

- Massaging mixed-signed objectives
- Recognizing a min cut problem



$$c(A', B') = \sum_{i \in A' \setminus \{s\}} b_i + \sum_{j \in B' \setminus \{t\}} a_j + \sum_{\{i,j\} \in E : |A' \cap \{i,j\}| = 1} p_{ij}.$$

Learning Goal

- Project selection problem and its reduction to min cut
- Turning hard constraints into punishments in the objective

Motivating Problem

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- Formally, choose a set $S \subseteq P$ that maximizes $\sum_{i \in S} p_i$, such that $R_i \subseteq S$ for every $i \in S$

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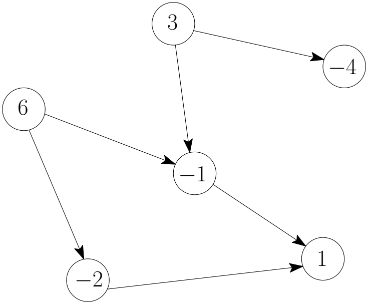

Adjusting the objective

- The objective is in fact mixed signed.
  - Let A be the set of projects with nonnegative profits, and B the set with negative profits, then the profit of any $S \subseteq P$ is

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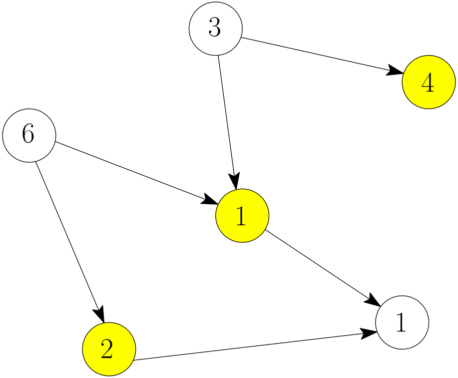
  - To reduce to min cut, we need a minimization problem:

$$\min_{S \subseteq P} \sum_{i \in A \setminus S} p_i + \sum_{i \in S \cap B} |p_i|.$$

An illustration



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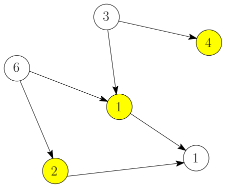
- How about the prerequisite requirements?
- If no edge in G is in the cut we output, all prerequisite requirements are satisfied!
- Let all the original edges in G carry infinite (or large enough) capacity.

Illustration of flow network



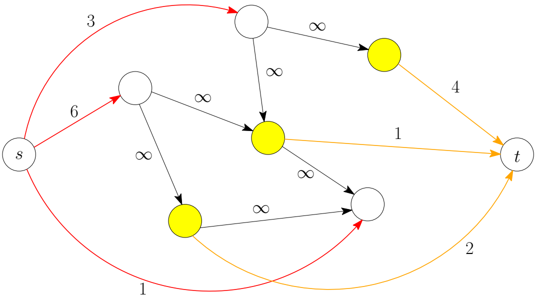
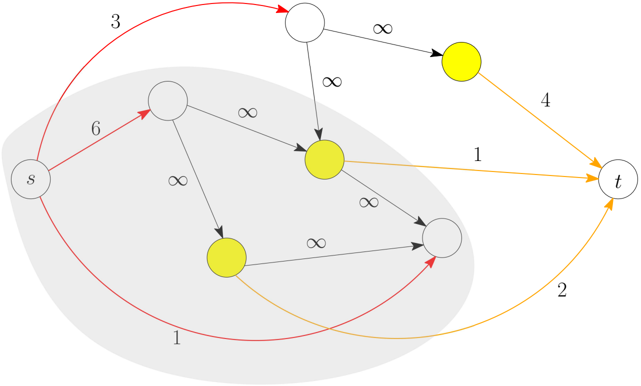


Illustration of a cut



Last remark

- A commonly used idea when doing reductions among problems: convert “hard” constraints to “soft” ones. I.e., if in one problem, certain patterns are forbidden, then in the other problem, punish such patterns in the objective — when the punishment is high enough, such patterns are forbidden from the solution.

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- (Outside this course:) Recall Lagrangian multipliers from calculus. Hard constraints are softened into punishment in the objective, and the multipliers adjust how heavy the punishment is.