Learning Goals

- Random variables and their expectations
- Expectation of some distributions (Indicator variables/Bernoulli, binomial, geometric)
- Linearity of expectations
- Analyze two examples: guessing cards and coupon collection
- Analyze probability of correctness of simple randomized algorithms, as exemplified by MAX 3-SAT

Random variables

- Random variables map the sample space to real numbers $X : \Omega \to \mathbb{R}$.
  - Examples in this class all have random variables taking nonnegative integer values.

Random variables

- Random variables map the sample space to real numbers $X : \Omega \to \mathbb{R}$.
  - Examples in this class all have random variables taking nonnegative integer values.
- Let X be a random variable on a probability space, for a number j,

$$\mathbf{Pr}[X = j] := \mathbf{Pr}[\{\omega \in \Omega : X(\omega) = j\}].$$

Random variables

- Random variables map the sample space to real numbers $X : \Omega \to \mathbb{R}$.
  - Examples in this class all have random variables taking nonnegative integer values.
- Let X be a random variable on a probability space, for a number j,

$$\mathbf{Pr}\left[X = j\right] := \mathbf{Pr}\left[\{\omega \in \Omega : X(\omega) = j\}\right].$$

- Example: Toss a dice, let X be the result (number of pips). Then $\forall i \in \{1, 2, \cdots, 6\}$, $\mathbf{Pr}[X = i] = 1/6$.

Random variables

- Random variables map the sample space to real numbers $X : \Omega \to \mathbb{R}$.
  - Examples in this class all have random variables taking nonnegative integer values.
- Let X be a random variable on a probability space, for a number j,

$$\mathbf{Pr}\left[X = j\right] := \mathbf{Pr}\left[\{\omega \in \Omega : X(\omega) = j\}\right].$$

- Example: Toss a dice, let X be the result (number of pips). Then $\forall i \in \{1, 2, \cdots, 6\}$, $\mathbf{Pr}[X = i] = 1/6$.
- Example: For an event A, let X be 1 if A happens, and 0 if not. Then $\mathbf{Pr}[X = 1] = \mathbf{Pr}[A]$.
  - X is called the *indicator variable* of A.
  - A random variable that only takes values 0 or 1 is said to be drawn from a *Bernoulli distribution*.

Expectation of a random variable

- The *expectation* of a random variable X is

$$\mathbf{E}[X] := \sum_{j=0}^{\infty} j \cdot \mathbf{Pr}[X = j].$$

Expectation of a random variable

- The *expectation* of a random variable X is

$$\mathbf{E}[X] := \sum_{j=0}^{\infty} j \cdot \mathbf{Pr}[X = j].$$

- Example: If X is the indicator variable of event A, then $\mathbf{E}[X] = \mathbf{Pr}[X = 1] = \mathbf{Pr}[A]$.

Expectation of a random variable

- The *expectation* of a random variable X is

$$\mathbf{E}[X] := \sum_{j=0}^{\infty} j \cdot \mathbf{Pr}[X = j].$$

- Example: If X is the indicator variable of event A, then $\mathbf{E}[X] = \mathbf{Pr}[X = 1] = \mathbf{Pr}[A]$.
- Example: If X is the result of a die toss, then

$$\mathbf{E}[X] = \frac{1}{6}\sum_{i=1}^{6} i = \frac{7}{2}.$$

$$\mathbf{E}\left[X^2\right] = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{91}{6}.$$

Note $\mathbf{E}[X^2] \neq (\mathbf{E}[X])^2$.

Example

- Example: Toss a coin that shows Heads with probability p. Keep tossing until one sees a head. Let X be the number of tosses.

Example

- Example: Toss a coin that shows Heads with probability p. Keep tossing until one sees a head. Let X be the number of tosses.

$$\mathbf{Pr}\left[X=j\right]=p(1-p)^{j-1};$$

$$\mathbf{E}\left[X\right]=\sum_{j=1}^{\infty} j\,\mathbf{Pr}\left[X=j\right]=\sum_{j=1}^{\infty} jp(1-p)^{j-1}=p\cdot\frac{1}{p^2}=\frac{1}{p}.$$

Example

- Example: Toss a coin that shows Heads with probability p. Keep tossing until one sees a head. Let X be the number of tosses.

$$\mathbf{Pr}\left[X = j\right] = p(1-p)^{j-1};$$

$$\mathbf{E}\left[X\right] = \sum_{j=1}^{\infty} j\,\mathbf{Pr}\left[X = j\right] = \sum_{j=1}^{\infty} jp(1-p)^{j-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

For $x < 1$,

$$\sum_{j=1}^{\infty} jx^{j-1} = \sum_{j=1}^{\infty} (x^j)' = \left(\sum_{j=1}^{\infty} x^j\right)' = \left(\frac{1}{1-x} - 1\right)' = \frac{1}{(1-x)^2}.$$

Linearity of expectation

- For random variables X and Y defined on the same probability space, a new random variable $X + Y$ is given by
  $(X + Y)(\omega) = X(\omega) + Y(\omega)$ for any sample point ω.

Linearity of expectation

- For random variables X and Y defined on the same probability space, a new random variable $X+Y$ is given by $(X+Y)(\omega) = X(\omega) + Y(\omega)$ for any sample point ω.
- Example: Toss two dices. Let X be the result of the first dice, and Y that of the second. Then $X+Y$ is the random variable for the sum of the two results.

Linearity of expectation

- For random variables X and Y defined on the same probability space, a new random variable $X+Y$ is given by $(X+Y)(\omega) = X(\omega) + Y(\omega)$ for any sample point ω.
- Example: Toss two dices. Let X be the result of the first dice, and Y that of the second. Then $X+Y$ is the random variable for the sum of the two results.

Theorem

For any collection of random variables $X_1, \cdots, X_n$ (defined on the same probability space),

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}\left[X_i\right].$$

Linearity of expectation

- For random variables X and Y defined on the same probability space, a new random variable $X + Y$ is given by $(X + Y)(\omega) = X(\omega) + Y(\omega)$ for any sample point ω.
- Example: Toss two dices. Let X be the result of the first dice, and Y that of the second. Then $X + Y$ is the random variable for the sum of the two results.

Theorem

For any collection of random variables $X_1, \cdots, X_n$ (defined on the same probability space),

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}\left[X_i\right].$$

Diversion: Independence among random variables

Definition

Two random variables are *independent* if for any i, j, the events $X = i$ and $Y = j$ are independent.

Remark

Linearity of expectation does NOT need independence among the random variables!

Examples of linearity of expectations: Guessing cards

Shuffle a deck of n distinct cards, and reveal them one by one. Before each revelation, make a uniformly random guess. How many guesses are correct in expectation?

Examples of linearity of expectations: Guessing cards

Shuffle a deck of n distinct cards, and reveal them one by one. Before each revelation, make a uniformly random guess. How many guesses are correct in expectation?

- Let X_i be the indicator variable for the i-th guess being correct, then $\mathbf{E}[X_i] = 1/n$.

Examples of linearity of expectations: Guessing cards

Shuffle a deck of n distinct cards, and reveal them one by one. Before each revelation, make a uniformly random guess. How many guesses are correct in expectation?

- Let X_i be the indicator variable for the $i^{\text{-th}}$ guess being correct, then $\mathbf{E}[X_i] = 1/n$.
- The total number of correct guesses is $X := \sum_{i=1}^{n} X_i$. So $\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[X_i] = n \cdot \frac{1}{n} = 1$.

Guessing cards (continued)

What if in the i-th round, we guess a card uniformly at random among the cards that haven't shown up?

Guessing cards (continued)

What if in the i-th round, we guess a card uniformly at random among the cards that haven't shown up?

- Let Y_i be the indicator variable for the i-th guess being correct. Then $\mathbf{E}[Y_i] = \frac{1}{n-i+1}$.

Guessing cards (continued)

What if in the i-th round, we guess a card uniformly at random among the cards that haven't shown up?

- Let Y_i be the indicator variable for the i-th guess being correct. Then $\mathbf{E}[Y_i] = \frac{1}{n-i+1}$.
- The total number of crrect guess is $Y := \sum_i Y_i$. So

$$\mathbf{E}[Y] = \sum_{i=1}^{n} \mathbf{E}[Y_i] = \sum_{i=1}^{n} \frac{1}{n-i+1} = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n.$$

Examples of linearity of expectations: Coupon collection

A coffee shop gives you, for any puchase of coffee, one of n different coupons uniformly at random. After you collect all n coupons, you get a free cup. How many cups do you expect to buy before you get a free one?

Examples of linearity of expectations: Coupon collection

A coffee shop gives you, for any puchase of coffee, one of n different coupons uniformly at random. After you collect all n coupons, you get a free cup. How many cups do you expect to buy before you get a free one?

- Let X_i be the number of purchases we make to get the $i^{\text{-th}}$ coupon *after* we've collected $i - 1$ coupons.

Examples of linearity of expectations: Coupon collection

A coffee shop gives you, for any puchase of coffee, one of n different coupons uniformly at random. After you collect all n coupons, you get a free cup. How many cups do you expect to buy before you get a free one?

- Let X_i be the number of purchases we make to get the $i^{\text{-th}}$ coupon *after* we've collected $i-1$ coupons.
- There are $n-i+1$ unseen coupons, and the probability we see one of them in each purchase is $\frac{n-i+1}{n}$.

Examples of linearity of expectations: Coupon collection

A coffee shop gives you, for any puchase of coffee, one of n different coupons uniformly at random. After you collect all n coupons, you get a free cup. How many cups do you expect to buy before you get a free one?

- Let X_i be the number of purchases we make to get the i-th coupon *after* we've collected $i-1$ coupons.
- There are $n-i+1$ unseen coupons, and the probability we see one of them in each purchase is $\frac{n-i+1}{n}$.
- $\mathbf{E}[X_i] = \frac{n}{n-i+1}$ (from the earlier example about tossing coins.)

Examples of linearity of expectations: Coupon collection

A coffee shop gives you, for any puchase of coffee, one of n different coupons uniformly at random. After you collect all n coupons, you get a free cup. How many cups do you expect to buy before you get a free one?

- Let X_i be the number of purchases we make to get the i-th coupon *after* we've collected $i-1$ coupons.
- There are $n-i+1$ unseen coupons, and the probability we see one of them in each purchase is $\frac{n-i+1}{n}$.
- $\mathbf{E}[X_i] = \frac{n}{n-i+1}$ (from the earlier example about tossing coins.)
- Therefore the expected total number of purchases is

$$\sum_{i=1}^{n} \frac{n}{n-i+1} = n \cdot \sum_{i=1}^{n} \frac{1}{i} \approx n \ln n.$$

Recipe for expectation calculation

- Express the quantity we are interested in as a random variable
- Express the random variable as a sum of random variables whose expectations are easy to compute
- Apply linearity of expectation (without worrying about independence)!

Algorithmic application: MAX 3-SAT

- MAX 3-SAT problem: Given a 3-SAT formula with n variables and m clauses, find a truth assignment that satisfies as many clauses as possible.

Algorithmic application: MAX 3-SAT

- MAX 3-SAT problem: Given a 3-SAT formula with n variables and m clauses, find a truth assignment that satisfies as many clauses as possible.
- The problem is obviously NP-hard.

Algorithmic application: MAX 3-SAT

- MAX 3-SAT problem: Given a 3-SAT formula with n variables and m clauses, find a truth assignment that satisfies as many clauses as possible.
- The problem is obviously NP-hard.
- A randomzied algorithm: let each variable be TRUE with probability $\frac{1}{2}$, independently.

Algorithmic application: MAX 3-SAT

- MAX 3-SAT problem: Given a 3-SAT formula with n variables and m clauses, find a truth assignment that satisfies as many clauses as possible.
- The problem is obviously NP-hard.
- A randomzied algorithm: let each variable be TRUE with probability $\frac{1}{2}$, independently.

Claim

Under the uniformly random truth assignment, in expectation $\frac{7}{8}m$ clauses are satisfied.

MAX 3-SAT Analysis

Claim

Under the uniformly random truth assignment, in expectation $\frac{7}{8}m$ clauses are satisfied.

Proof.

Let X_i be the indicator variable for the i-th clause to be satisfied, then the number of satisfied clauses is $\sum_i X_i$.

MAX 3-SAT Analysis

Claim

Under the uniformly random truth assignment, in expectation $\frac{7}{8}m$ clauses are satisfied.

Proof.

Let X_i be the indicator variable for the i-th clause to be satisfied, then the number of satisfied clauses is $\sum_i X_i$.

$$\mathbf{E}\left[\sum_i X_i\right] = \sum_i \mathbf{E}[X_i] = \sum_{i=1}^{m} \mathbf{Pr}[\text{clause } i \text{ is satisfied}] = \frac{7}{8}m.$$

□