Pf. ($\forall$ s-t cut, $f(A, B) = |f|$.)

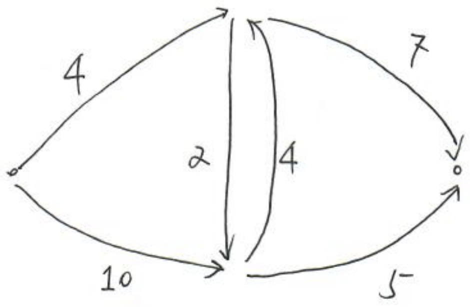
We prove by inducting on $|A|$. The base case when $A = \{s\}$ is the definition of $|f|$.

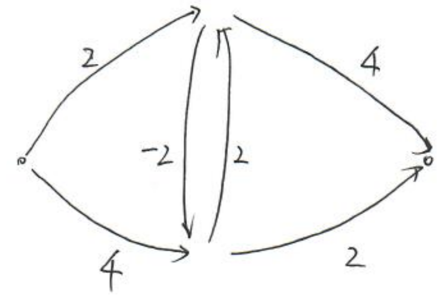
Induction step: Assuming $f(A, B) = |f|$, we prove $f(A \cup \{v\}, B - \{v\}) = f$ for any $v \in B \setminus \{t\}$. Consider $f(A, B) - f(A \cup \{v\}, B - \{v\})$. Any edge (u, v) where $u \in A$ is in the cut (A, B) and not in $(A \cup \{v\}, B - \{v\})$; any edge ~~$(u, v)$~~ (v, u) where $u \in B \setminus \{v\}$ is not in (A, B) but in $(A \cup \{v\}, B - \{v\})$. These are the only differences between the two cuts. Therefore

~~|f|~~ $$f(A, B) - f(A \cup \{v\}, B - \{v\}) = -\sum_{u \in A} f(u, v) + \sum_{u \in B \setminus \{v\}} f(v, u)$$

$$= -\sum_{u \in A} f(u, v) - \sum_{u \in B \setminus \{v\}} f(u, v) = \sum_{u \neq v} f(u, v) = 0.$$

Therefore $f(A \cup \{v\}, B - \{v\}) = |f|$ as well. ☑





Pf. 1). If f' is a flow on G_f

then $\forall (u,v)$, $(f+f')(u,v) = f(u,v) + f'(u,v)$

$= -f(v,u) - f'(v,u) = -(f+f')(v,u)$

for conservation of flow:

$\forall v \notin \{s,t\}$, we have $\sum_{u \notin v} (f+f')(u,v)$

$= \sum_{u} f(u,v) + \sum_{u} f'(u,v)$

$= 0 + 0 = 0.$

For capacity: $\forall (u,v)$,

$(f+f')(u,v) = f(u,v) + f'(u,v)$

$\leq f(u,v) + r(u,v)$

$= f(u,v) + c(u,v) - f(u,v)$

$= c(u,v).$

Pf. ③ → ② :

If there's no augmenting path, it means the set of nodes reachable from s in G_f is not V.

Let A be <u>these</u> nodes. $B = V - A$. $\Rightarrow t \in B$.

then there's no edge from A to B in G_f.

which means $r(u,v) = 0$, $\forall u \in A$, $v \in B$

$$r(u,v) = c(u,v) - f(u,v).$$

$$\Rightarrow f(A,B) = c(A,B)$$

$$f(A,B) = |f|.$$

② → ① : $\forall f'$, $|f'| \leq c(A,B)$.

① → ③ : Suppose there's an augmenting path in G_f. since the capacities of all edges along this path are positive, the minimum among these, "bottleneck", Δ is also positive. But Δ amount of flow can be sent from s to t in G_f. But f plus this flow is another flow in G, which has value Δ more than $|f|$. $\Rightarrow \Leftarrow$

Pf. Perfect matching $\Rightarrow$ condition. obvious.

If there's perfect matching, s.t. $\forall u \in U$ is matched to $\varphi(u)$. then, $\forall S \subseteq U$,

$$\delta(S) \supseteq \{ \varphi(u), u \in S \}.$$

$$|\delta(S)| \geq |\{\varphi(u), u \in S\}| = |S|.$$

$\Leftarrow$: Consider the flow construction.

By theorem, $\exists$ perfect matching $\Leftrightarrow$ the min cut has size n.

Consider any s-t cut A, B. Suppose $A \cap U = S$, then w.l.g. all neighbors of S are in A as well (because each of them, if in B, contributes at least a capacity of 1 by virtue of them being neighbors of S)

The other nodes in V, w.l.g. belong to B.

Now the capacity of this cut is

$$|\delta(S)| + |U - S| = |\delta(S)| + n - |S| \geq n.$$

Proof of König-Evergáry Thm.

As we explained in class, any vertex cover has at least as many vertices as edges in any matching. (This is true even for general graphs.) We therefore only need to show that for a bipartite graph, the size of a maximum matching is at least the size of a minimum vertex cover. Consider again the flow construction for bipartite matching. By the max flow min cut theorem, we only need to show that any cut in the network has capacity more than or equal to ~~any~~ the size of a minimum vertex cover. To that end, consider any s-t cut (A, B). Let S be $A \cap U$. By the same argument before, we may w.l.o.g. assume ~~B~~ $A \cap V = \delta(S)$, where $\delta(S)$ denotes the neighbors of nodes in S. The capacity of the cut is $|\delta(S)| + |U-S|$, (as we have shown in the proof of Hall's Theorem). But observe that $\delta(S) \sqcup (U - S)$ is a vertex cover in the original graph. (Any edge either has its endpoint in U belonging to $U-S$, or, if not, its other endpoint must be a neighbor of S, i.e. in $\delta(S)$.)

Therefore, $|\text{any cut}| \geq |\text{a vertex cover}| \geq |\text{min vertex cover}|$.

$$\Rightarrow |\text{max matching}| = |\text{min cut}| \geq |\text{min vertex cover}|.$$

Pf (of running time assuming Lemmas)

length of augmenting path $\le n$.

For each length, there can be $\le m$ rounds.

Each round takes time $O(m)$ { residual graph construction, augmenting path finding (BFS) }

$O(m^2 n)$ in total.

Claim. The shortest distance from s to a node in the residual graph is equal to the layer numbes of the node in the level graph of the residual graph. (minus 1).

Proof of Lemma 1. In each round of the algorithm, one of the forward edges in the level graph becomes saturated and removed. ~~Some edges become backward.~~ Some backward edges may be added. The only possible addition to the layes graph is that some sideways and backward edges may become forward, but that happens precisely when the endpoints of such edges become more distanced from the source. (i.e. theip move to a deeper layes in the level graph.

Proof of Lemma 2. When a node moves to a deeper layer, it will not appear in any shortest augmenting path until t also moves to a deeper layer. Therefore until that happens, the shortest augmenting path can only use forward edges in the original level graph. These edges disappear one by one and s becomes disconnected from t after at most m rounds.

Assume team A wins all future matches, ending with H ~~wo~~ wins.

If w_i denotes the number of past wins for team i, then A has a chance to win the champion only if all other teams end with $\leq H$ wins,

$\Leftrightarrow$ each team i ~~can~~ wins $\leq H - w_i = C_i$ times in the future.

Now the problem is:

can we decide ~~an~~ outcomes for the matches (each of which generates one winner) so that no team i exceeds its capacity C_i.

This is an assignment problem, assigning each match to one of the 2 teams involved, while respecting each team's capacity constraint.

Graph as on the board.

<u>Claim</u>. Team A can win iff the max flow in the graph saturates all edges from s to the match nodes.

$\Leftrightarrow$ The cut $(\{s\}, V - \{s\})$ is a min cut of the graph. (V = nodes)

Consider any ~~a~~ minimum cut ~~(A, B)~~ (B, D) whose capacity is strictly smaller than # future matches.

Suppose $S = B \cap$ Team nodes.

$\bar{S} = D \cap$ Team nodes.

then <u>all the matches involving any teams in $\bar{S}$</u> must be in D as well; also, all the matches that only involve teams in S are in B as well.

The capacity of the cut is

$$\sum_{i \in S} C_i + \left| \text{all matches involving any team in } \bar{S} \right|$$

$$< \# \text{ matches}$$

$$\sum_{i \in S} C_i < \left| \text{matches played among } S \right|.$$