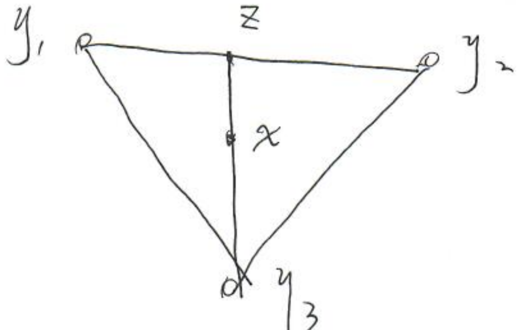
Pf. { If $S_1, \cdots, S_n$ are convex, then $\bigcap_{i=1}^{n} S_i$ is convex.

Take $x, y \in \bigcap_{i=1}^{n} S_i$, ~~the~~ $\forall \lambda \in [0,1]$,

we know $\lambda x + (1-\lambda) y \in S_i$, $\forall i$.

$$\Rightarrow \lambda x + (1-\lambda) y \in \bigcap_{i=1}^{n} S_i$$



$$z = \lambda_1' y_1 + (1-\lambda_1') y_2.$$

$$x = \lambda_3 y_3 + (1-\lambda_3) z$$

$$= \lambda_3 y_3 + (1-\lambda_3)(\lambda_1' y_1 + (1-\lambda_1')) y_2)$$

$$= \lambda_1' (1-\lambda_3) y_1 + (1-\lambda_1')(1-\lambda_3) y_2 + \lambda_3 y_3$$

Pf (Corollary).

By the lemma, any feasible solution ~~x~~ is a convex combination of vertices. In particular, let x^* be ~~the~~ an optimal solution ~~x~~, then

$$x^* = \sum_{i=1}^{k} \lambda_i \underline{\underline{v^i}} \quad \text{for some } \lambda_1, \cdots, \lambda_k,$$

$\sum \lambda_i = 1$, $v^1, \cdots, v^k$ are vertices of the polytope.

Let the linear function be $f: x \mapsto y^T x$.

$$f(x^*) = y^T \cdot \sum_{i=1}^{k} \lambda_i v^i$$

$$= \sum_{i=1}^{k} \lambda_i (y^T v^i)$$

$$= \sum_{i=1}^{k} \lambda_i \, f(v^i).$$

Because of optimality of x^*, $f(x^*) \geq f(v^i)$, $\forall i$.

$\Rightarrow$ for any i s.t. $\lambda_i > 0$, $f(v^i) = f(x^*)$.

(If not, then $\sum_{i' \neq i} \lambda_{i'} f(v^{i'}) + \lambda_i f(v^i)$

$$\leq (1-\lambda_i) f(x^*) + \lambda_i f(v^i)$$

$$< f(x^*).$$)

Pf. If ~~conditions~~ vertex, then condition satisfied.

① by def.

② Prove by contradiction. Suppose $A_{i_1} \cdots A_{i_k}$ is not full rank, then $\exists \vec{y}$ s.t. $A_{i_j}^T y = 0$, $j = 1, \cdots, k$.

Consider $\vec{x} + \varepsilon \vec{y}$ and $x - \varepsilon \vec{y}$ for small enough $\varepsilon > 0$.

For $j = 1, \cdots, k$, $A_{i_j}^T (x \pm \varepsilon \vec{y}) = b_{i_j} + \varepsilon A_{i_j}^T \vec{y} = b_{i_j}$

For any other constraint $A_j^T x \leq b_j$.

$A_j^T (x \pm \varepsilon \vec{y})$ ~~$\leq b_j + \varepsilon A_j^T \vec{y} \leq b_j$.~~

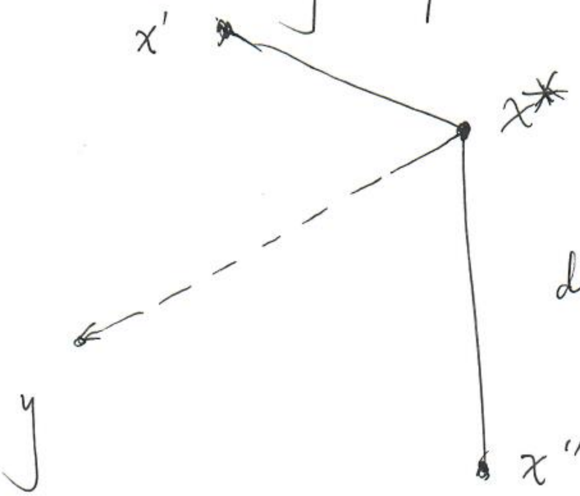
$\Delta_j > 0$.

$= b_{i_j} - \Delta_j + \varepsilon A_j^T \vec{y} \leq b_{i_j}$ as long as $\varepsilon \leq \frac{\Delta_j}{|A_j^T y|}$ for small $\varepsilon > 0$.

Thm (Simplex).

Suppose Simplex finds x^*, and y is a global optimal s.t. $\vec{c}^T \vec{y} > \vec{c}^T x^*$.

then we know moving in the direction of $\vec{y} - \vec{x}^*$ strictly impsoves the objective value.



by some algebraic/geometric argument, one can show that the direction of $\vec{y} - x^*$ is a convex combinations of the directions along adjacent edges of x^*.

Simliar arguments as before show that moving along one of theses edges must improve the objective.

Duality:

multiply by each constraint $A^i x \le b_i$ by a factor $y_i \geq 0$, to obtain $\sum_i y_i A^i x \le \sum y_i b_i$

If for each x_j, <u>the coefficient in the sum is $\ge c_j$</u>.

then $\sum_j c_j x_j \le \sum_i y_i A^i x \le \sum y_i b_i$.

Proof (Complementary Slackness)

The theorem states that $\vec{y}^*$ being optimal dual solution

$$\Rightarrow \vec{b}^T \vec{y}^* = (A\vec{x}^*)^T \vec{y}^* \text{ for any optimal primal solution } \vec{x}^*.$$

By dual feasibility, $\vec{y}^{T*}(A\vec{x}^*) = (\vec{y}^{*T}A)\vec{x}^* \geq \vec{c}^T\vec{x}^*$

By ~~strong duality~~ primal feasibility, $\vec{y}^{*T}(A\vec{x}^*) \leq \vec{y}^{*T}\vec{b}$

By strong duality, $\vec{y}^{*T}\vec{b} = \vec{c}^T\vec{x}^*$

Therefore, $\vec{c}^T\vec{x}^* \leq (\vec{y}^{*T}A)\vec{x}^* = \vec{y}^{*T}(A\vec{x}^*) \leq \vec{y}^{*T}\vec{b} = \vec{c}^T\vec{x}^*$

$$\Rightarrow \vec{y}^{*T}(A\vec{x}^*) = \vec{y}^{*T}\vec{b}.$$