Suppose $\mathcal{M}$ is not a matroid,

then $\exists\ A, B \in \mathcal{M}$, $|A| < |B|$,

$\nexists\ a_i \in B - A$ s.t. $A \sqcup \{a_i\} \in \mathcal{M}$.

<u>Claim</u>. W.L.O.G., B is maximal in $\mathcal{M}$.

<u>Pf</u>. If B is not maximal, then

$\exists\ a_j \in E$, s.t. $B \sqcup \{a_j\} \in \mathcal{M}$.

Let $B' = B \sqcup \{a_j\}$.

If $A \sqcup \{a_j\} \in \mathcal{M}$, $A' = A \sqcup \{a_j\}$,

o.w. $A' = A$.

$|A'| < |B'|$.

Also, $\nexists\ a_i \in B' - A'$, s.t. $A' \sqcup \{a_i\} \in \mathcal{M}$.

~~$a_j \notin B' - A'$.~~ Suppose not,

$a_i \in B - A$, and $A' \sqcup \{a_i\} \in \mathcal{M}$.

by downward closure, $A \sqcup \{a_i\} \in \mathcal{M}$. $\Rightarrow\Leftarrow$

Applying this procedure repeatedly, we can get a B that's maximal.

Case 1: If A is not maximal,

$$w(a_i) = \begin{cases} 0, & a_i \in A. \\ 1, & a_i \in B - A \\ 2n^2. & a_i \notin A \cup B. \end{cases}$$

Greedy will output a strict superset of A, with weight $\geq 2n^2$. But OPT $= B$. with weight $|B-A| < 2n^2$.

Case 2: If A is maximal. $|B| = b + |A \cap B|$, $|A| = a + |A \cap B|$ Because $|B| > |A|$.

Let $a = |A - B|$, $b = |B - A|$. $b > a$.

$$w(a_i) = \begin{cases} 0, & a_i \in A \cap B. \\ a+1, & a_i \in B - A, \\ b+1, & a_i \in A - B, \\ 2n^2, & a_i \notin A \cup B. \end{cases}$$

Greedy outputs B, with weight $\underline{\underline{b}} \cdot (a+1) = ab + b$.

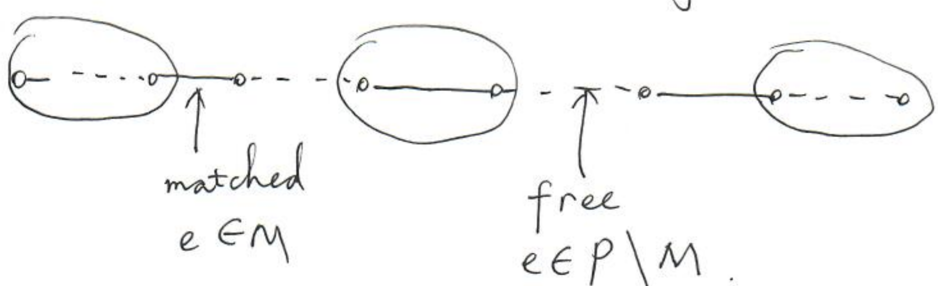
Optimal solution is A, w/weight $a \cdot (b+1) = ab + a$.

But $ab + a < ab + b$.

Pf. (1) $M \oplus P$ is still a matching.

~~Edges in~~ Nodes outside P ~~are~~ still have degree 0 or 1.

So we consider only nodes in P.



After taking the symmetric difference, the endpoints of P have degree 1.

the nodes in the interior of P have degree 2 in P, and exactly one of the 2 edeges incident to such a node is in M; hence in $P \oplus M$, these nodes have degree 1.

Hence all nodes have degree 1 or 0.

So $M \oplus P$ is a matching.

Compared w/ M, $M \oplus P$ have 2 fewer free nodes (the endpoints of P). hence

$$|M \oplus P| = |M| + 1.$$

Pf. We need to prove: if M is a matching but not of maximum cardinality, then it admits an augmenting path.

We know $\exists M^*$: M^* is a matching, and $|M^*| > |M|$.

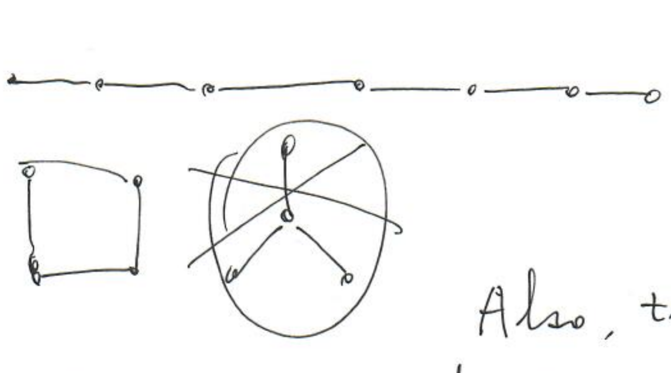
Take $M^* \oplus M$.

if an edge is in $M \cap M^*$, then $e \notin M^* \oplus M$.

Consider possible degrees of a node in $M^* \oplus M$.

This degree can be only 0, 1 or 2.

(Each node can have ~~an incident~~ at most one incident in M, and ≤ 1 incident edge in M^*. Hence in $M^* \oplus M$ any node can have ≤ 2 incident edges)



The graph can only contain (vertex-disjoint) cycles and paths.

Also, these paths and cycles are alternating.

Because any node in a path of degree 2 has exactly one edge from M and one other from M^*. Therefore the path can not have two edges sharing a node s.t. both are from M or both are from M^*.

But there must exist an alternating path, because otherwise $|M^* - M| = |M - M^*|$, which contradicts $|M^*| > |M|$.

Among the alternating paths, we must have ≥ 1 that starts and ends with an edge in M^*.

Only paths that start and end w/ ~~an~~ edges in M^* could explain the size difference btwn M^* & M.

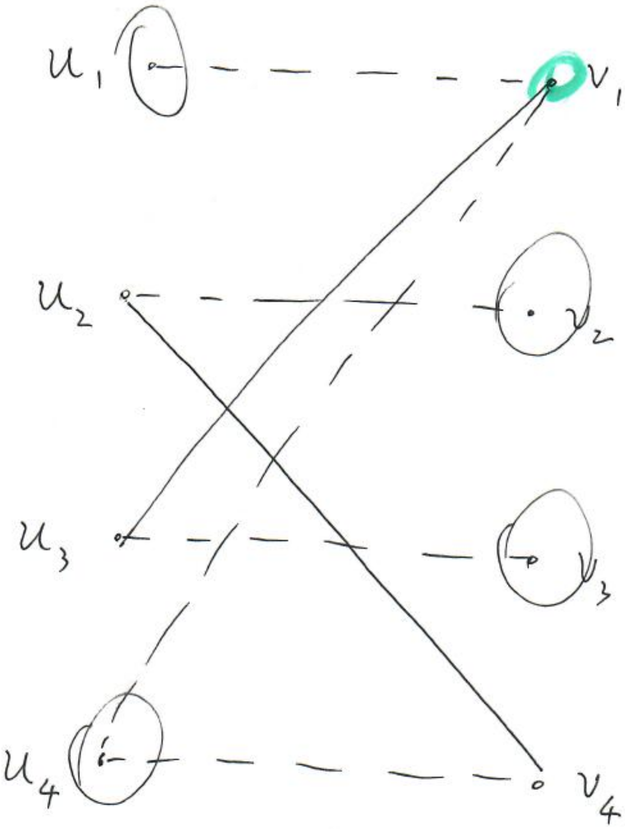
But such a path is augmenting w.r.t. M.

Pf.: Given an augmenting path in $G=(U,V,E)$. because it starts and ends w/ free vertices, and is alternating, its two endpoints are in U and V respectively. ~~So~~ Let its starting point be in V. Then the first edge in the path goes from V to U, and the second goes from U to V and is in M. By definition of $D(G,M)$, this second edge is oriented from U to V. The 1^{st}, 3^{rd}, 5^{th}, ...; edges ~~go from V to U,~~ ~~and~~ are free, and therefore are oriented from V to U; 2^{nd}, 4^{th}, ..., edges are matched, and are oriented in $D(G,M)$ from U to V. As one traces from the starting point to the ending points one goes along the ~~a~~ orientation of each edge as in $D(G,M)$, and the endpoints are in $F \cap V$ and $F \cap U$, respectively.

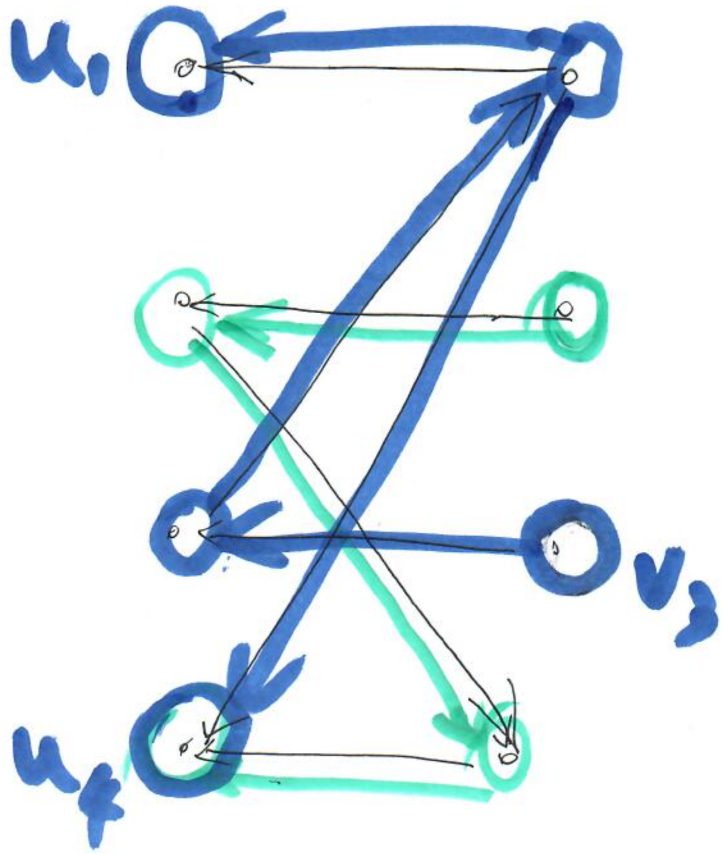
Given a path from $V \cap F$ to $U \cap F$ in $D(G, M)$, only need to show its edges are alternating in G. Since its edges have to ~~alternating~~ alternate btwn $V \to U$ and $U \to V$, they are alternating btwn free and matched. $\boxtimes$

Pf. First, assume there are ~~$k \geq k$~~ k vertex-disjoint augmenting paths. let the minimum length among them be l, then $lk \leq n$. $\Rightarrow l \leq \frac{n}{k}$.
(Pigeonhole principle)

For the first part, take M^* to be a maximum matching, then $|M^*| = |M| + k$.
Look at $M^* \oplus M$, ~~then~~ it is a vertex-disjoint union of alternating paths and alternating cycles.
① Alternating cycles and paths that start w/ free node and end w/ matched node contain the same # edges from M and M^*.



u_1
v_1
u_2
v_2
u_3
v_3
u_4
v_4



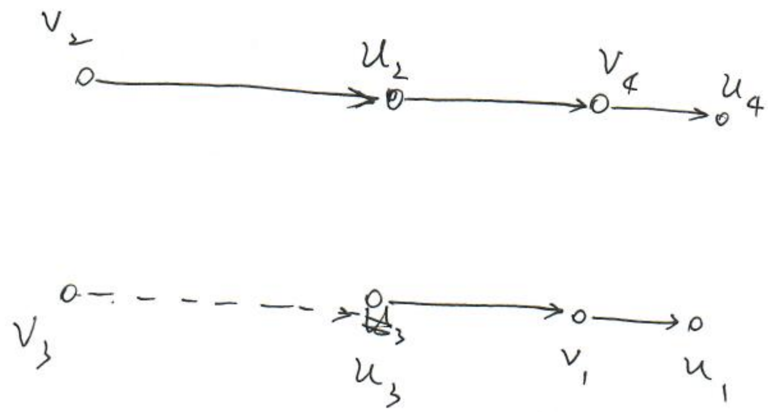
u_1
v_3
u_4

② Alternating path that start and end with edges in $M \setminus M^*$ don't exist in $M^* \oplus M$.

③ Augmenting paths for M (those that start and end w/ edges in $M^* \setminus M$).

Each of the augmenting path for M contains one more edge in $M^* \setminus M$ than in $M \setminus M^*$.

So there are k of them, and they are vertex disjoint. $\boxtimes$



Hungarian tree:

① Do BFS from $F \cap V$, but keep ~~in~~ on each node a counter for # incoming edges ~~going into~~.

The first layer of the "tree" are free nodes.

② ~~Start~~

If there free nodes in deeper levels. they are in $F \cap U$.

Start with the ~~sm~~ level of minimum depth ($\neq 1$) with free nodes, and backtrak the edges until we reach the first layer. This gives an augmenting path whose length is the #layer-1.

For each node on this path: ③ Put it on a deletion queue

For each node in the queue, deleta, and subtract the counter of its children by 1.

If any child's counter becomes 0, put the child on the deletion queue.

Repeat until the queue is empty.

~~Then~~

Repeat the procedure for the same layer until it has no free nodes.

Hopcroft & Karp Alg.

$M = \emptyset$

1. Construct Hungarian Tree w.r.t. M.
2. Find a block set of augmenting paths ($O(m)$)
3. Augment M with these paths.

Repeat, until no augmenting can be found.

Correctness: By the end of the alg, no augmenting path exists for M.

Pf (of Corollary).

After $\sqrt{n}$ rounds, by the Lemma, the shortest augmenting path is of length $\Omega(\sqrt{n})$. ($\geq 2\sqrt{n}-1$).

This means the difference btwn the matching's size and the maximum matching is $\leq O(\sqrt{n})$.

($\leq \sqrt{n}$).

(Because if $\exists M^*$, s.t. $|M^*| - |M| > \sqrt{n}$, then by lemma, $\exists$ augmenting path of length $\leq \frac{n}{\sqrt{n}} = \sqrt{n}$)

Therefore after this step ($\sqrt{n}$), the alg will continue for no more than $\sqrt{n}$ loops.

Pf of ~~Lemmaa~~ Lemma using Lemma

~~A~~ In each round of the alg., we augment M by a blocking set; all these paths are of minimum length l w.r.t. M. In the next round, ~~we find~~ any augmenting path q we can find must ~~have an e~~ have ~~e~~ either a longer length or ~~have~~ a non empty intersection ~~of~~ with one of the paths in the previous blocking set.

(If q does not intersect w/ any previous round augmenting paths, it must be also an augmenting path for M;)

But if q intersects a p in the previous blocking set, by last Lemma, $|q| \geq |p| + 2|p \cap q| \geq p + 2$.

Pf of final lemma

If $q \cap p = \emptyset$, then q is an aug. path for M.

$|q| \geq |p|$. So assume $q \cap p \neq \emptyset$.

Look at $p \oplus q$. We want to show that there are ~~at least~~ 2 vertex-disjoint augmenting paths (w.r.t. M) in $p \oplus q$. Then $|p \oplus q| \geq 2|p|$

$$\cancel{|q|}\ \cancel{|p|}\ |p| + |q| = |p \oplus q| + 2|p \cap q| \geq 2|p| + 2|p \cap q|$$

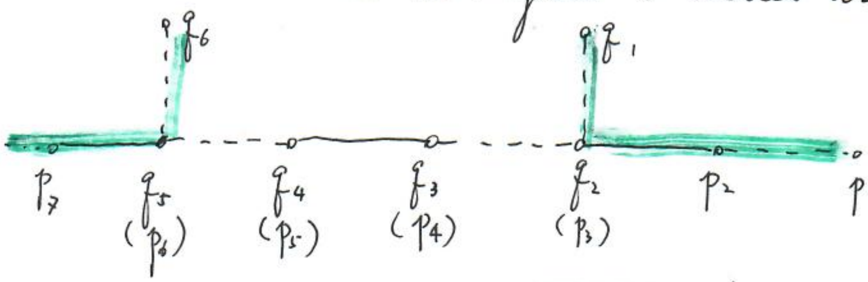
$$|q| \geq |p| + 2|p \cap q|.$$

Proof that $p \oplus q$ contains two augmenting paths w.r.t. M.

We make the following observations:

1. There are 4 free nodes in $p \oplus q$.
2. Each node in $p \overset{\sqcup}{\cancel{\oplus}} q$ has degree ≤ 3.

   The ~~et~~ degree 3 nodes are where q enters or leaves p:



: free edge (w.r.t. M)

: matched edge.

p_i : nodes in p

q_i : nodes in q.

: edge in $p \oplus q$.

In this example, p_6/q_5 and p_3/q_2 have degree 3 in $p \sqcup q$.

Hence a node in $p \oplus q$ has degree ≤ 2, since each node with degree 3 in $p \sqcup q$ has an edge in $p \cap q$.

3. q enters p after a free edge, and leaves p before a free edge.

   In the example above, (q_1, q_2) ~~is~~ and (q_5, q_6) are free.

   (This is because the nodes at the ~~ad~~ intersection (q_2/p_3 and p_6/q_5 in the example) are matched in p; they cannot have another matched edge in $q - p$; that would violate M being a matching.)

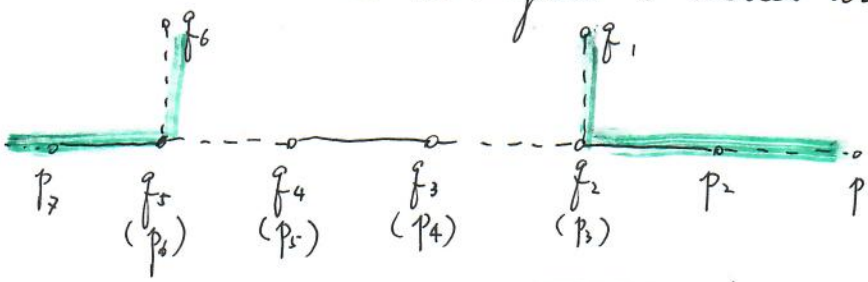
Therefore $p \oplus q$ contains alternating cycles and alternating paths, and 4 ~~not~~ free nodes; there must be 2 augmenting paths.

Proof that $p \oplus q$ contains two augmenting paths w.r.t. M.

We make the following observations:

1. There are 4 free nodes in $p \oplus q$.
2. Each node in $p\ \cancel{\oplus} \sqcup\ q$ has degree ≤ 3.

   The ~~ed~~ degree 3 nodes are where q enters or leaves p:



- - - - : free edge (w.r.t. M)

—— matched edge.

p_i : nodes in p

q_i : nodes in q.

▬▬ : edge in $p \oplus q$.

In this example, p_6/q_5 and p_3/q_2 have degree 3 in $p \sqcup q$.

Hence a node in $p \oplus q$ has degree ≤ 2, since each node with degree 3 in $p \sqcup q$ has an edge in $p \cap q$.

3. q enters p after a free edge, and leaves p before a free edge.

   In the example above, (q_1, q_2) ~~is~~ and (q_5, q_6) are free.

   (This is because the nodes at the ~~ed~~ intersection (q_2/p_3 and p_6/q_5 in the example) are matched in p; they cannot have another matched edge in $q - p$; that would violate M being a matching.)

Therefore $p \oplus q$ contains alternating cycles and alternating paths, and 4 ~~not~~ free nodes; there must be 2 augmenting paths.