Problem Set 2

Due date: Feb 9, 2018 at 9pm

1. (3 pts) Given two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ defined on the same universe U, the intersection of the two matroids is the set of their common independent sets, i.e., $\{S \subseteq U | S \in \mathcal{M}_1, S \in \mathcal{M}_2\}$.

   Given a bipartite graph G, show that the set of matchings in it is the intersection of two matroids. (In other words, this question asks you to define two matroids and show that their intersection is the set of matchings in the graph.)

2. (5 pts) (Exercise 7.22 from Kleinberg & Tardos) Given an n by n matrix, we denote by m_{ij} the entry in the i-th row and j-th column. By *swap rows i and j*, we swap the values of m_{ik} and m_{jk} for $k = 1, 2, \ldots, n$. Swapping columns is defined analogously. These two kinds of operations are collectively called *swapping operations*. The diagonal entries are m_{ii}, for $i = 1, \ldots, n$.

   A matrix with each entry equal to either 0 or 1 is said to be *rearrangeable* if there is a sequence of swapping operations after which the resulting matrix has all its diagonal entries equal to 1. Give a polynomial time algorithm to decide if a matrix with 0-1 entries is rearrangeable.

3. (5 pts) (Exercise 7.26 from Kleinberg & Tardos) Consider the following simplified model of a cellular phone network in a sparsely populated area. There are n base stations and n phones, and we are their locations $b_1, \ldots, b_n$, and $p_1, \ldots, p_n$, respectively, as points in a plane. A base station can serve a phone if their straight line distance is within a given parameter Δ. At any point, each base station can at most one phone. Suppose a particular phone i is going to travel a distance of z towards east while all other phones remain stationary. Give an algorithm running in time $O(n^3)$ that determines if it is possible to keep all phones connected during this travel. If this is possible, your algorithm should also output a sequence of assignments that maintain full connectivity; if it is not possible, your algorithm should report a location of phone i where it is impossible to have all phones connected.

4. (5 pts) Recall that a basis of a matroid is an independent set of maximum cardinality. Given two bases A and B, from the exchange property we know that, for every element $a \in A$, there exists an element $b \in B - \{A - \{a\}\}$ such that $A - \{a\} \cup \{b\}$ is another basis. We show something stronger in this problem: given any two bases A and B of a matroid, there is a *one-to-one* mapping φ between A and B, such that for element a in A, $A - \{a\} \cup \{\varphi(a)\}$ is independent. Note that, since $|A| = |B|$, the mapping is in fact a bijection.

   (Hint: Try constructing a bipartite graph whose nodes set are A and B, respectively, with appropriately defined edges, then try to apply Hall's theorem.)