Learning Goals

- State the purpose of Bloom filters
- Understand the tradeoff between space and accuracy in Bloom filters
- Analyze the performance of a Bloom filter

Membership Checking

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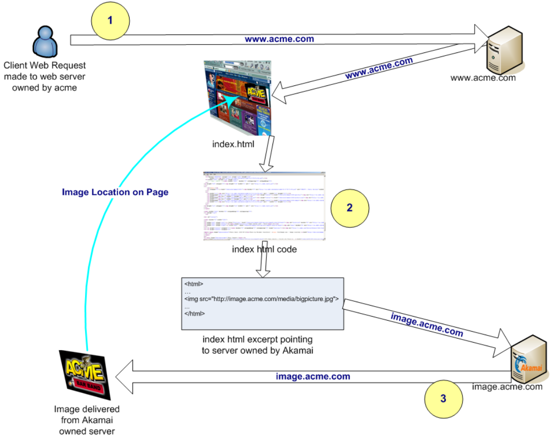
Illustration of Content Delivery Network (CDN)



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<html>
...
<img src="http://image.acme.com/media/bigpicture.jpg">
...
</html>
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Credit: Kim Meyrick — http://en.wikipedia.org/wiki/Image:Akamaiprocess.png

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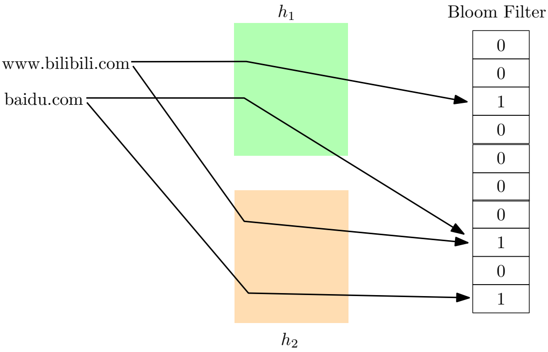
- For every entry $x \in S$, mark $B[h_1(x)] = \cdots = B[h_k(x)] = 1$.
- When checking the membership of a key x, return "$\textsc{Yes}$" if $B[h_1(x)] = \cdots = B[h_k(x)] = 1$; if any of these is 0, return "$\textsc{No}$".

Illustration



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- By symmetry this is minimized at $p = \frac{1}{2}$, so $k = \ln 2 \cdot (m/n)$.
- The false positive rate is roughly $(1/2)^{\ln 2 \cdot (m/n)} \approx (0.61850)^{m/n}$.

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- The idea of increased efficiency at the cost of some fault toleration is a recurring theme in handling with big data.
- This clever use of hash functions will also reappear later in the course.