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- Again, we must use space $O(\log d, \frac{1}{\epsilon})$.

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- $\mathrm{Var}[X_{(1)}] = \frac{\ell}{(\ell+1)^2(\ell+2)} \leq \frac{1}{(\ell+1)^2}$.
- We can apply the Chebyshev bound, although the variance is a bit too large for our purpose.

Flajolet and Martin

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  - Use real hash functions. Discretize the range. Possibly use k-universal hash family.
  - The minimum of $h(i_t)$ tends to be voltaile: a single bad event ruins the estimate.
  - To make the estimate more stable, we can keep track of the t minimum hash values instead of one.

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  - If $|S| < t$, return $|S|$.
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$\mathrm{Var}[Y] = \mathbf{E}[Y^2] - \mathbf{E}[Y]^2 \leq \mathbf{E}[Y^2] = \mathbf{Pr}[Y = 1]$. $\square$

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  - For any pair of indices, they are mapped to the same address with probability $\frac{1}{D}$.
  - There are $\binom{\ell}{2}$ pairs, so the probability that any clash happens is $\binom{\ell}{2} \cdot \frac{1}{D} \leq \frac{1}{d}$.

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- By pairwise independence we have $\mathrm{Var}[Z] = \sum_i \mathrm{Var}[Z_i] \leq t$.

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- The optimal algorithm uses space $O(\log d + \epsilon^{-2})$!