# Learning Goals

- Understand the desiderata of hashing
- State the guarantee of universal hashing
- Understand the construction of universal hashing using finite fields

# The Fundamental Data-Structuring Problem

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- There may be other operations such as , DELETE, MERGE, TRAVERSE, etc..

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  - Example: when $U$ is the set of all URLs.
- Opening an array for all possible key values is wasteful and often impractical.
- Using a linked list means very slow (linear time) $\textsc{Find}$ operation.

# Idea of Hashing

Open an array (*hash table*) with size $m$, with $m \ll |U|$. Have a function $h : U \to \{0, 1, \ldots, m-1\}$, and store entry $i$ at position $h(i)$.

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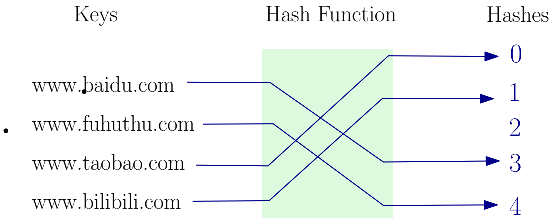

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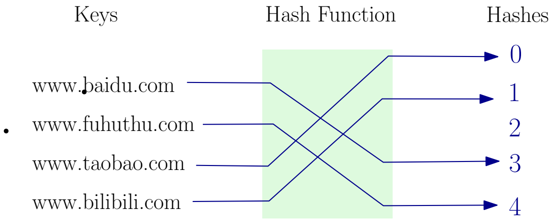


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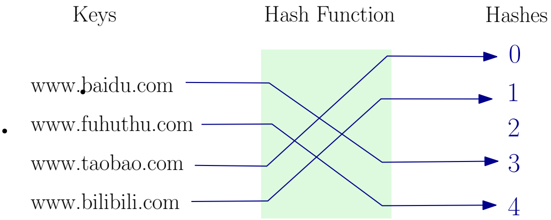


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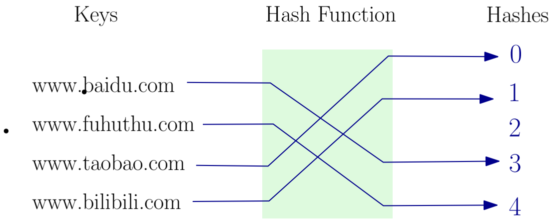


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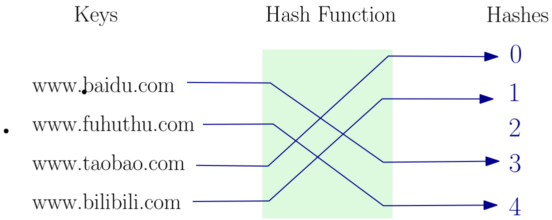


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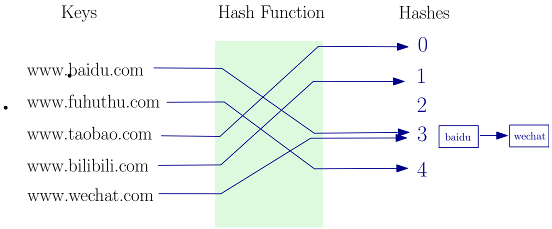
- What if two keys are mapped to the same position? I.e., if for $x, y \in U, x \neq y, h(x) = h(y)$.
  - This is called a *collision*.

# Dealing with collisions

Separate chaining: build a linked list at each entry of the hash table.

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- Ideally $m = O(n)$.
- There should be few collisions.

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**Proposition**

For any given function $h : U \to \{0, 1, \ldots, m-1\}$, if $|U| \geq (n-1)m+1$, then there exists $S \subseteq U$, with $|S| = n$, whose elements are mapped to the same position by $h$.

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- Let $h$ be a function that maps any key to a position uniformly at random?
  - How do you keep the record of $h$ and access that record? We are back to where we started!
  - This amounts to choosing at random from a family of $m^{|U|}$ hash function.

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**Definition**

A family $H$ of hash functions is *universal* if for any $x \neq y$ in $U$,

$$\mathbf{Pr}_{h \leftarrow H}\left[h(x) = h(y)\right] \leq \frac{1}{m},$$

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**Remark**

The property is not assuming the keys are themselves randomly chosen from $U$! This is still worst case guarantee.

# Guarantees of Universal Hashgin

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Let $H$ be a universal hash functions family from $U$ to $\{0, \ldots, m-1\}$ and let $h$ be a hash function uniformly sampled from $H$. For *any* set $S \subseteq U$ with $|S| = n$ and any $x \in S$, the expected number of collisions of $x$ under $h$ is at most $n/m$.

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**Proof.**

For $x \neq y$ in $S$, let $I_{xy}$ be the indicator variable for the event that $h(x) = h(y)$, then $\mathbf{E}[I_{xy}] = \mathbf{Pr}[h(x) = h(y)] \leq \frac{1}{m}$.

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- As long as $m = \Omega(n)$, the expected number of collisions is $O(1)$. Total time for FIND is $O(1)$.

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$$\forall x \in U, h_{\mathbf{r}}(x) := r_1 x_1 + \ldots + r_k x_k \mod m.$$

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For any choice of $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k$, there is a unique $r_i$ that solves (1). With probability $1/m$, $r_i$ is chosen to be that. $\square$