Learning Goals

- State the condition Markov inequality
- Understand distributions for which Markov inequality is tight
- Define perfect hashing
- Implementation and proof of perfect hashing
- Understand the method of amplification by independent trials

Concentration Inqualities

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- Tools that upper bound the probability with which a random variable deviates far from its expectation are known as *concentration inequalities* or *tail bounds*.

Markov Inequality

Theorem (Markov Inequality)

If X is a random variable that takes nonnegative value with probability 1, then for any $\alpha > 1$,

$$\mathbf{Pr}\left[X \geq \alpha\,\mathbf{E}\left[X\right]\right] \leq \frac{1}{\alpha}.$$

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Proof.

Let Y be the indicator variable for $X \geq \alpha\, \mathbf{E}[X]$. Then

$$\mathbf{Pr}\left[X \geq \mathbf{E}\left[X\right]\right] = \mathbf{Pr}\left[Y = 1\right] = \mathbf{E}\left[Y\right] \leq \mathbf{E}\left[\frac{X}{\alpha\, \mathbf{E}[X]}\right] = \frac{1}{\alpha}.$$

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    - Stated this way, the inequality has bite only for $a > \mathbf{E}[X]$.
- Note the condition that X must be a nonnegative random variable.

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- The distribution for which Markov inequality tight is a two-point distribution.
- With this intuition, it is not difficult to prove the following corollary:

Corollary (Reverse Markov Inequality)

If X is a random variable that is never larger than a, then for any $b < a$,

$$\mathbf{Pr}\left[X \leq b\right] \leq \frac{a - \mathbf{E}[X]}{a - b}.$$

Application: Perfect Hashing

Definition

A hash function $h : U \to \{0, \ldots, m-1\}$ is *perfect* on $S \subseteq U$ if $\textsc{Find}(x)$ for every $x \in S$ takes $O(1)$ time.

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- It does not follow immediately that there exists an $h \in H$ under which every element has only $O(1)$ collisions.
  - In fact, we will see next week that, under the mapping that sends every element in U uniformly at random to $\{0, \ldots, m-1\}$, for $m = n$, with high probability the worst bucket has $\Theta(\log n / \log\log n)$ collisions.

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$\mathbf{Pr}_{h\sim H}[h(x) = h(y)] \leq \frac{1}{m}$.
By the union bound, the probability that any collision happens is at most $\sum_{x\neq y\in S} \frac{1}{m} < \frac{n^2}{2} \cdot \frac{1}{m} \leq \frac{1}{2}$. □

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  - Let $A[\cdot]$ be the array for the first level hash, and h be a hash function from U to $\{0, \ldots, n-1\}$.
  - For each $i = 0, \ldots, n-1$, let n_i be the number of collisions in that bucket. Set up a hash table B_i of size n_i^2, and a *perfect* hash function mapping U to $\{0, \ldots, n_i^2 - 1\}$.

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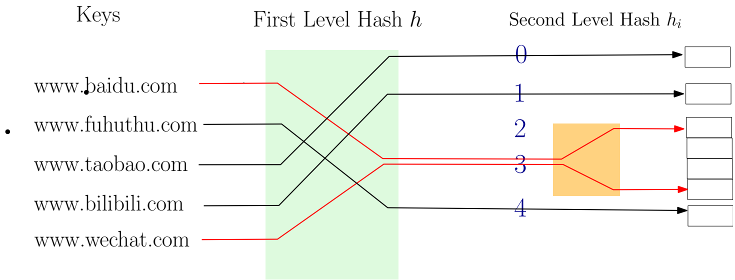
  - When looking up x, we first find its position in the first level. Let j be $h(x)$. Then we look up $B_j[h_j(x)]$.

Illustration: Perfect Hashing



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Lemma

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Proof.

Game plan: we first show that $\mathbf{E}[\sum_i n_i^2]$ is no more than $2n$. Then the conclusion follows from Markov inequality.

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For $x \ne y$ in S, let C_{xy} be the indicator variable for the event that x clashes with y under h, then $\mathbf{E}[C_{xy}] \le \frac{1}{n}$ by universality.

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(Let S_i be $h^{-1}(i)$, then $\sum_{x \in S_i} \sum_{y \in S \setminus \{x\}} C_{xy}$ contributes $n_i(n_i - 1)$.)

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Now we can bound

$$\mathbf{E}\left[\sum_{x \in S} \sum_{y \in S \setminus \{x\}}\right] C_{xy} \leq n(n-1) \cdot \frac{1}{n} \leq n.$$

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Key observation: $\sum_i n_i^2 = n + \sum_{x\in S}\sum_{y\in S\setminus\{x\}} C_{xy}$.
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Therefore $\mathbf{E}[\sum_i n_i^2] \le 2n$. □

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- We can check if we succeed in polynomial time. If not, we simply try again.
- After k trials, we succeed with probability $1 - \frac{1}{2^k}$.