Learning Goals

- Concept of dimensenality reduction
- Correctly state the procedure and guarantee of Johnson-Lindenstrauss transform
- Proof idea of JL-transform

Dimensionality Reduction

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- Many algorithms are very slow when run on high dimensional input
  - *Curse of dimensionality*
- *Dimensionality reduction*: Transform data to lower dimensions while preserving information useful for analysis/application

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  For $x, y \in \mathbb{R}^d$, $||x - y||$ is their ℓ_2-distance, or Euclidean distance.
- The *Johnson-Lindenstrauss* transform is a *randomized* dimensionality reduction algorithm that *approximately* preserves Euclidean distances.

JL Statement

Theorem (Johnson-Lindenstrauss)

For arbitrary $x_1, \ldots, x_n \in \mathbb{R}^d$, *and any* $\epsilon \in (0,1)$, *there is* $t = O(\log n/\epsilon^2)$ *such that there are* $y_1, \ldots, y_n \in \mathbb{R}^t$ *with*

$$(1-\epsilon)||x_j|| \leq ||y_j|| \leq (1+\epsilon)||x_j||, \quad \forall j$$
$$(1-\epsilon)||x_j - x_{j'}|| \leq ||y_j - y_{j'}|| \leq (1+\epsilon)||x_j - x_{j'}||, \quad \forall j, j'.$$

Moreover, $y_1, \ldots, y_n$ *can be computed in polynomial time.*

Main Lemma

Lemma

Distributional JL For any $\epsilon, \delta \in (0, 1]$, *there is a* $t = O(\log(1/\delta)/\epsilon^2)$ *and a random* *linear* *map* $f : \mathbb{R}^d \to \mathbb{R}^t$, *such that, for any* $v \in \mathbb{R}^d$ *with* $||v|| = 1$,

$$\mathbf{Pr}\left[1 - \epsilon \leq \frac{||f(v)||}{\sqrt{t}} \leq 1 + \epsilon\right] \geq 1 - 2\delta.$$

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Proof of Theorem using Lemma.

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$$\mathcal{E}_w := \left\{\frac{||f(w)||}{\sqrt{t}} \notin [1 - \epsilon, 1 + \epsilon] \cdot ||w||\right\} = \left\{\frac{||f(v)||}{\sqrt{t}} \notin [1 - \epsilon, 1 + \epsilon]\right\}.$$

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Each such event has probability $\leq 2\delta$. By union bound, the probability that none of these happen is $\leq |W| \cdot 2\delta \leq \frac{2}{n}$. □

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- A random variable is drawn from *Gaussian distribution* (or *Normal distribution*) $\mathcal{N}(\mu, \sigma^2)$ if its PDF is

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  - In particular, the *standard normal distribution* has PDF $\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$.
  - If $X \sim \mathcal{N}(0,1)$, then $\sigma X + \mu \sim \mathcal{N}(\mu, \sigma^2)$.

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Proof of Theorem.

We show only the zero mean case. For $X \sim \mathcal{N}(0, \sigma^2)$,

$$\begin{aligned} \mathbf{E}\left[e^{\lambda X}\right] &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2} + \lambda x\right) \mathrm{d}x \\ &= \frac{e^{\sigma^2\lambda^2/2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma} - \sigma\lambda\right)^2} \mathrm{d}x = e^{\frac{\sigma^2\lambda^2}{2}}. \end{aligned}$$

So for independent $X \sim \mathcal{N}(0, \sigma_1^2)$, $Y \sim \mathcal{N}(0, \sigma_2^2)$,
$\mathbf{E}[e^{\lambda(X+Y)}] = \mathbf{E}[e^{\lambda X}] \cdot \mathbf{E}[e^{\lambda Y}] = e^{(\sigma_1^2 + \sigma_2^2)\lambda^2/2}$. □

Idea of JL

- For $x \in \mathbb{R}^d$ with $||x|| = 1$, let $G_1, \cdots, G_d$ be i.i.d. from $\mathcal{N}(0, 1)$, then $\sum_i G_i x_i \sim \mathcal{N}(0, ||x||^2) = \mathcal{G}(0, 1)$.

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  - Let $A' = \frac{1}{\sqrt{t}} A$, then $\mathbf{E}[||A'x||^2] = 1$.
- We just need to show that the empirical average converges to the expectation fast enough with t.

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$$\mathbf{Pr}\left[\frac{||Ax||}{\sqrt{t}} \geq (1+\epsilon)\right] = \mathbf{Pr}\left[Y \geq (1+\epsilon)^2 t\right]$$
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Let's bound $\mathbf{Pr}[Y > \alpha]$ for any α. For any $\lambda > 0$, we have

$$\mathbf{Pr}\left[Y \geq \alpha\right] = \mathbf{Pr}\left[e^{\lambda Y} \geq e^{\lambda \alpha}\right] \leq \frac{\mathbf{E}[e^{\lambda Y}]}{e^{\lambda \alpha}} = \frac{\prod_i \mathbf{E}[e^{\lambda Y_i^2}]}{e^{\lambda \alpha}}.$$

Moment Generating Function of χ^2-distributions

If $X_1, \cdot, X_k$ are independent standard normal random variables, then $Q = \sum_i X_i$ is said to be distributed according to the χ^2-distribution with k degrees of freedom.

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$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-2\lambda}} \int e^{-y^2/2}\, \mathrm{d}y = \frac{1}{\sqrt{1-2\lambda}}.$$

where we substituted $y = \sqrt{1-2\lambda}x$. □

Finishing Proof of Lemma

$$\mathbf{Pr}\left[Y \geq \alpha\right] \leq \frac{\prod_i \mathbf{E}[e^{\lambda Y_i^2}]}{e^{\lambda\alpha}} = (1 - 2\lambda)^{-t/2} e^{-\lambda\alpha}.$$

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Now let α be $(1+\epsilon)^2 t$, we get

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Using basic calculus, we can show $\ln(1+\epsilon) \leq \epsilon - \frac{\epsilon^2}{4}$ for $\epsilon \in [0, 1]$, so we have

$$\mathbf{Pr}\left[Y \geq (1+\epsilon)^2 t\right] \leq e^{-\frac{3}{4}\epsilon^2 t}.$$