Learning Goals

- Understand the design idea of skip lists
- Carry out more involved probabilistic runtime analysis using Chernoff bound and union bound
- Understand the idea of SkipNet in Peer-to-Peer systems

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    - One level above, at L_2, we have a linked list storing every four node from L_0, or every other node from L_1, also sorted, with $\lfloor n/4 \rfloor$ nodes, etc..
- Each copy of the node in L_i stores pointers to its copies in L_{i-1} and L_{i+1} (if they exist), and also the nodes the precede and follow it in L_i.

Skip List: Illustration

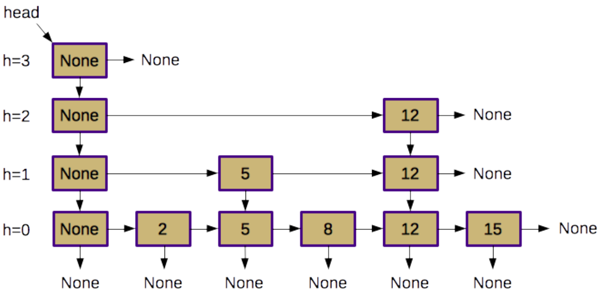

Image credit: Mike Lam at James Madison University

Find in Skip List

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- In actual implementation, we may store only the keys in levels other than L_0, and store the actual content only in nodes of L_0.
- Problem: INSERT and DELETE are combersome.

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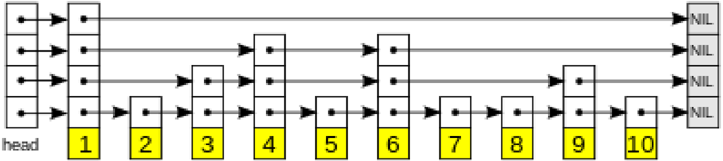
- We just need to show that this randomized construction yields similar performance for Find as the previous deterministic structure.

Randomized Skip List: ILlustration



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Apply Chernoff Bound

Take X to be, say, $36 \log n$, and let Y_i, $i = 1, \cdots, X$, be the indicator variable that the i-th step is upward. Then $\mathbf{E}[Y_i] = \frac{1}{2}$. Let Y be $\sum_i Y_i$.

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By Chernoff bound,

$$\begin{aligned}\mathbf{Pr}\left[Y \leq 3\log n\right] &= \mathbf{Pr}\left[Y \leq \mathbf{E}\left[Y\right] - 15\log n\right] \\ &\leq \exp\left(-\frac{2\cdot(15\log n)^2}{36\log n}\right) < \frac{1}{n^2}.\end{aligned}$$

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This analysis was performed for a specific node x. By the union bound, with probability at least $1 - \frac{1}{n}$, no node takes more than $36 \log n$ steps to reach level H.

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We have bounded $\mathbf{Pr}[A] \leq \frac{1}{n^2}$, and $\mathbf{Pr}[B] \leq \frac{1}{n}$.
Now by a final union bound, with probability at least $1 - \frac{2}{n}$, there are no nodes beyond level $L_{3 \log n}$ and every node reaches that level within $36 \log n$ steps. So Find takes time $O(\log n)$ for every node with high probability.

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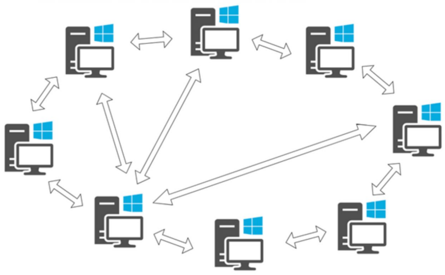

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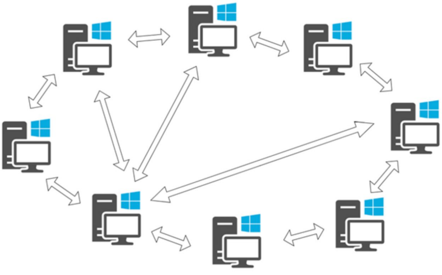

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- A request of a node to communicate with another can take $O(n)$ time to traverse the network if we are not careful.

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- To access a node, we go as far as possible on a high level, then descend and continue.