Learning Goals

- Definition of a Treap and its motivating ideas
- Definition of a Heap
- Implementation of Treap insertion
- Analysis of the expected performance of a Treap

Treap: Motivating Ideas

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- A binary search tree's shape depends on the arrival order of the nodes.
  - If the nodes $1, 2, \ldots, n$ arrive in this increasing order and are added with the naïve BST INSERT, the resulting tree will be a linked list.
  - Intuitively, for less adversarial arrival orders, the tree should be somewhat balanced.
- In fact, it can be shown that, if the nodes arrive in a uniformly random order, the expected height of the resulting BST is $O(\log n)$.

Proof Sketch for Randomly Built BST

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- A common trick to deal with this is to say $\mathbf{E}[\max\{h_{i-1}, h_{n-i}\}] \leq \mathbf{E}[h_{i-1} + h_{n-i}] = \mathbf{E}[h_i] + \mathbf{E}[h_{n-i}]$.

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- But here such a bound would be too loose.
    - If we expect h_i and h_{n-i} differ not much, then we'd lose a factor of 2 each time we apply this.

Proof Sketch for Randomly Built BST (Cont.)

- Another clever idea: we can amplify the quantity we are interested in, so that the estimation error caused by $\mathbf{E}[\max\{h_{i-1}, h_{n-i}\}] \leq \mathbf{E}[h_{i-1}] + \mathbf{E}[h_{n-i}]$ is more negligible.

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- This is a tractable recursion, and one can show that $\mathbf{E}[H_n]$ is polynomial in n. Therefore $\mathbf{E}[h_n]$ is $O(\log n)$.
  - The latter is another consequence of Jensen's inequality: $2^{\mathbf{E}[h_n]} \leq \mathbf{E}[2^{h_n}] = \mathbf{E}[H_n] = O(n^c)$.

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  - The same is true for each of the subtrees.
  - The resulting tree has the property that, for any two nodes x and y, if x is an ancestor of y, then $\pi(x) < \pi(y)$.

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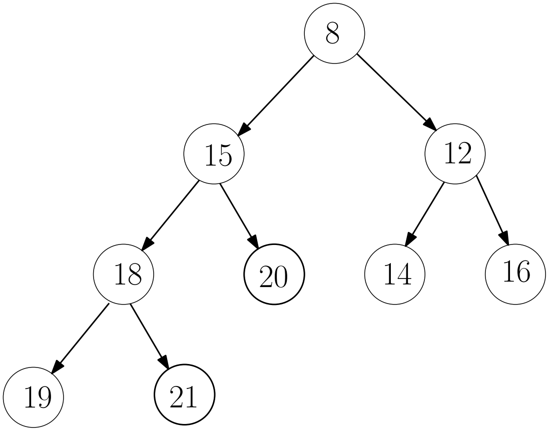
- Side remark: It is often useful to implement a heap in an array. One need not keep pointers for parents or children.

Heap: An Illustration



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- Operation $\textsc{Find}(x)$ is the same as in BST.
- Operation $\textsc{Insert}(x, r)$ first does the BST insertion using key values, and then assigns a uniformly random priority value to the new node and lets it swim (using tree rotations!) to restore the heap property on the priority values.

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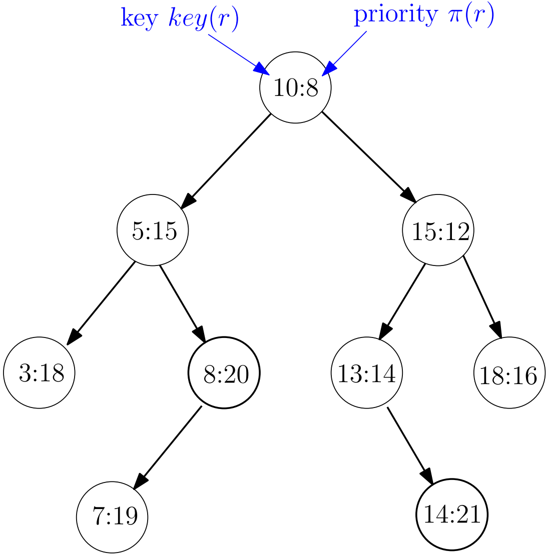

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- An alternative analysis of INSERT(x): it suffices to show that the ordinary tree insertion puts x at a leaf of depth $O(\log n)$.
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  - With probability $\frac{3}{4}$, the size of the subtree shrinks by a factor of $\frac{3}{4}$ when we go from $v_0 = r$ to v_1.
  - The same reasoning applies at every step down the path.

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- This is very similar to the analysis of Quicksort.
- An application of Chernoff bound shows that w.h.p. INSERT takes $O(\log n)$ time.
- In fact, another use of union bound shows that, w.h.p. the height of the Treap is $O(\log n)$.