
COMPRESSED SENSING

Hu Fu, Oct 2023

SHUFE

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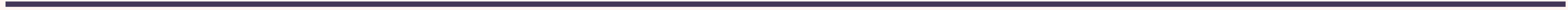
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 - We often have linear measurements of such signals
 - $\mathbf{y} = A\mathbf{x}$, where A is a matrix in $\mathbb{R}^{m \times n}$, with $m \ll n$
 - Given \mathbf{y} , can we recover \mathbf{x} ? We can design both the measurements A and the recovery algorithm.
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 - If \mathbf{x} is s -sparse, with high probability $\tilde{\mathbf{x}}$ is an **exact** recovery
 - Count Sketch consists of randomized linear measurements of \mathbf{x} . $\tilde{\mathbf{x}}$ is computed from them
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 - Pioneered by Candes & Tao

THE HALLMARK OF CS

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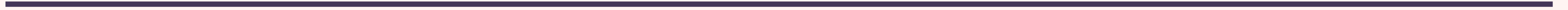
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 - This is solvable in polynomial time
 - Intuitively, why is the solution to this LP a good recovery of \mathbf{x} ?
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RESTRICTED ISOMETRY PROPERTY

- In order for (*) to produce good recoveries, we need A to approximately preserve ℓ_2 norms for all sparse vectors.

Def. A matrix $A \in \mathbb{R}^{m \times n}$ is said to satisfy the *restricted isometry property (RIP)* with parameters α, β and s if the inequality

$$\alpha \|\mathbf{v}\|_2 \leq \|A\mathbf{v}\|_2 \leq \beta \|\mathbf{v}\|_2$$

holds for all vectors $\mathbf{v} \in \mathbb{R}^n$ such that $\|\mathbf{v}\|_0 \leq s$

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Compare this with the
distributional JL lemma

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Theorem. Consider an $m \times n$ matrix A whose entries are i.i.d. drawn from the standard Gaussian $N(0,1)$. There are constants C and $c > 0$ such that, if $m \geq Cs \log(en/s)$, then with probability at least $1 - 2 \exp(-cm)$, the random matrix A satisfies the RIP with parameters $\alpha = 0.9\sqrt{m}$, $\beta = 1.1\sqrt{m}$ and s .

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The proof requires building up a bit of theory on random matrices.

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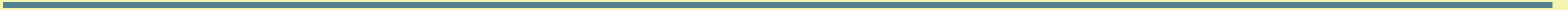
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Proof. $\|\mathbf{x}\|_1 = \|\hat{\mathbf{x}}\|_1 = \|\mathbf{x} + \mathbf{h}\|_1 = \|\mathbf{x}_{I_0} + \mathbf{h}_{I_0}\|_1 + \|\mathbf{h}_{\bar{I}_0}\|_1 \geq \|\mathbf{x}\|_1 - \|\mathbf{h}_{I_0}\|_1 + \|\mathbf{h}_{\bar{I}_0}\|_1$

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Sort coordinates of \mathbf{h} in \bar{I}_0 in absolute values. Let I_1 be the set of next largest λs coordinates, I_2 the next λs coordinates, and so on. Let $I_{0,1}$ be $I_0 \cup I_1$



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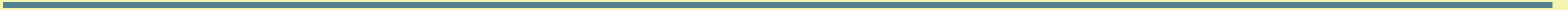
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Therefore $\sum_{i \geq 2} \|\mathbf{h}_{I_i}\|_2 \leq \frac{1}{\sqrt{\lambda s}} \sum_{i \geq 1} \|\mathbf{h}_{I_i}\|_1 = \frac{1}{\sqrt{\lambda s}} \|\mathbf{h}_{\bar{I}_0}\|_1 \leq \frac{1}{\sqrt{\lambda s}} \|\mathbf{h}_{I_0}\|_1 \leq \frac{1}{\sqrt{\lambda}} \|\mathbf{h}_{I_0}\|_2 \leq \frac{1}{\sqrt{\lambda}} \|\mathbf{h}_{I_{0,1}}\|_2$



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So we have $\frac{\beta}{\sqrt{\lambda}}\|\mathbf{h}_{I_{0,1}}\|_2 \geq \alpha\|\mathbf{h}_{I_{0,1}}\|_2$. But since $\lambda \geq (\beta/\alpha)^2$, this implies $\mathbf{h}_{I_{0,1}} = \mathbf{0}$, which by definition means $\mathbf{h} = \mathbf{0}$.
