Fast JL Transform

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        - Recall we had pairwise independent hash functions $h : [d] \to [w]$ and $g : [d] \to \{\pm 1\}$.
        - The operation of Count-Sketch can be seen as multiplying $\mathbf{x}$ by a $w \times d$ matrix M with $M_{h(i),i} = g(i)$ and all other entries 0.

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- In the worst case, $\mathbf{x}$ has only one non-zero entry, then t needs to be $\Theta(d)$ for us to see that entry
- Generally, this doesn't work well when $\dfrac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2} \approx 1$

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In fact in the original JL paper, the matrix is a projection onto a random t-dimensional subspace of $\mathbf{R}^d$

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  - Let $f = (|x_1^r|, \ldots, |x_d^r|),\ g = (1,\cdots,1), p = \dfrac{s}{r} \ge 1, q = \dfrac{s}{s-r}$.
  - We have $\|x\|_r^r \le \|x\|_s^r d^{\frac{s-r}{s}} \Rightarrow \|x\|_r \le \|x\|_s d^{\frac{1}{r}-\frac{1}{s}}$

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Proof. By induction, each entry of $H_{d/2}$ is in $\{\pm 1/\sqrt{d/2}\}$, so each entry of H_d is in $\{\pm 1/\sqrt{d}\}$.

$$H_d^\top H_d = \begin{pmatrix} H_{d/2}^\top & H_{d/2}^\top \\ H_{d/2}^\top & -H_{d/2}^\top \end{pmatrix} \begin{pmatrix} H_{d/2} & H_{d/2} \\ H_{d/2} & -H_{d/2} \end{pmatrix} /2 = \begin{pmatrix} 2H_{d/2}^\top H_{d/2} & 0 \\ 0 & 2H_{d/2}^\top H_{d/2} \end{pmatrix} /2 = I$$

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Proof. Let $T(d)$ be the time to compute $H_d \mathbf{x}$, then by recursive calls we have $T(d) = O(d) + 2T(d/2)$

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To bound $\|\mathbf{y}\|_\infty$, note that for each i, y_i has the same distribution as $\frac{1}{\sqrt{d}}\sum_j D_j x_j$ where D_j's are i.i.d. Rademacher variables. If we let $z_j := \frac{1}{\sqrt{d}} D_j x_j$, then $\mathbb{E}[z_j] = 0$. But Chernoff bound does not apply…

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Lemma. (Hoeffding's Lemma) If random variable X is in $[a_i, b_i]$, then $\mathbb{E}[e^{\lambda(X - \mathbf{E}[X])}] \leq \exp\left(\frac{\lambda^2(b-a)^2}{8}\right)$.

<u>Thm</u>. For nonzero $\mathbf{x} \in \mathbb{R}^d$, let $\mathbf{y} = HD\mathbf{x}$, then $\mathbb{P}\left[\frac{\|\mathbf{y}\|_\infty}{\|\mathbf{y}\|_2} \geq \sqrt{\frac{2\ln(4d/\delta)}{d}}\right] \leq \frac{\delta}{2}$.

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To bound $\|\mathbf{y}\|_\infty$, note that for each i, y_i has the same distribution as $\frac{1}{\sqrt{d}}\sum_j D_j x_j$ where D_j's are i.i.d.

Rademacher variables. If we let $z_j := \frac{1}{\sqrt{d}} D_j x_j$, then $\mathbb{E}[z_j] = 0$. Chernoff bound does not apply since

$z_j \in \{\frac{-x_j}{\sqrt{d}}, \frac{x_j}{\sqrt{d}}\}$.

To apply Hoeffding's bound, note that $\sum_j 4x_j^2/d^2 = 4/d$.

$$\mathbb{P}\left[|\sum_j z_j| \geq \sqrt{\frac{2\ln(4d/\delta)}{d}}\right] \leq 2\exp\left(-\frac{d}{2}\frac{2\ln(4d/\delta)}{d}\right) = 2 \cdot \frac{\delta}{4d} = \frac{\delta}{2d}.$$

The theorem follows from a union bound over y_i's.

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Claim. If $z_i^2 \leq \frac{2\ln(4d/\delta)}{d}$, for $t = \frac{2\ln^2(4d/\delta)\ln(4/\delta)}{\epsilon^2}$, $\mathbb{P}\left[\|\sqrt{\frac{d}{t}}S\mathbf{z}\|_2^2 \notin [1-\epsilon, 1+\epsilon]\right] \leq \frac{\delta}{2}$

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Another application of Hoeffding's bound.

Fast JL-Transform

Thm. For any $\mathbf{x} \in \mathbb{R}^d$ with $\|\mathbf{x}\| = 1$, for $t \geq \frac{2\ln^2(4d/\delta)\ln(4/\delta)}{\epsilon^2}$,

$$\mathbb{P}\left[\|\sqrt{\frac{d}{t}}SHD\mathbf{x}\| \in [1-\epsilon, 1+\epsilon]\right] \geq 1 - \delta.$$

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- $S \in \{0,1\}^{t\times d}$: Sampling matrix, with exactly one 1 in each row
- Running time: $O((d+t)\log d)$