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- Again, we must use space $O(\log d, \frac{1}{\epsilon})$.

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- $\mathrm{Var}[X_{(1)}] = \frac{\ell}{(\ell+1)^2(\ell+2)} \leq \frac{1}{(\ell+1)^2}$.
- We can apply the Chebyshev bound, although the variance is a bit too large for our purpose.

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  - The minimum of $h(i_t)$ tends to be voltaile: a single bad event ruins the estimate.
  - To make the estimate more stable, we may keep track of more than one smallest hash values.

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- By pairwise independence we have $\mathrm{Var}[Z] = \sum_i \mathrm{Var}[Z_i] \leq t$.

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- We have so far $\{\frac{tD}{X} > (1+\epsilon)\ell\} \Rightarrow \{Z \geq t\}$, $\mathbf{E}[Z] \leq (1-\frac{\epsilon}{2})t$, and $\mathrm{Var}[Z] \leq t$.

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Space usage:

- Storing the hash takes space $O(\log D) = O(\log d)$.

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Let Z be $\sum_{i=1}^{\ell} Z_i$, then $\mathbf{E}[Z] \ge (1+\frac{\epsilon}{2})t$, $\mathrm{Var}[Z] \le 2t$.
By Chebyshev inequality,

$$\mathbf{Pr}\left[\frac{tD}{X} < (1-\epsilon)\ell\right] \le \mathbf{Pr}\left[Z < t\right] \le \frac{\mathrm{Var}[Z]}{(\epsilon t/2)^2} \le \frac{8}{\epsilon^2 t} \le \frac{2\delta}{3}.$$

Combining everything, we have that with probability at least $1-\delta$, $|\frac{tD}{X} - \ell| \le \epsilon\ell$.
Space usage:

- Storing the hash takes space $O(\log D) = O(\log d)$.
- Storing S takes space $tO(\log D) = O(\frac{\log d}{\epsilon^2\delta})$.

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- The optimal algorithm uses space $O(\log d + \epsilon^{-2})$!