# Learning Goals

- State the condition Markov inequality
- Understand distributions for which Markov inequality is tight
- Define perfect hashing
- Implementation and proof of perfect hashing
- Understand the method of amplification by independent trials

# Concentration Inqualities

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- Such a phenomenon is called *concentration*.
- Tools that upper bound the probability with which a random variable deviates far from its expectation are known as *concentration inequalities* or *tail bounds*.

# Markov Inequality

**Theorem (Markov Inequality)**

*If $X$ is a random variable that takes nonnegative value with probability $1$, then for any $\alpha > 1$,*

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**Proof.**

Let $Y$ be the indicator variable for $X \geq \alpha\, \mathbf{E}[X]$. Then

$$\mathbf{Pr}\left[X \geq \mathbf{E}\left[X\right]\right] = \mathbf{Pr}\left[Y = 1\right] = \mathbf{E}\left[Y\right] \leq \mathbf{E}\left[\frac{X}{\alpha\, \mathbf{E}[X]}\right] = \frac{1}{\alpha}.$$

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  - Stated this way, the inequality has bite only for $a > \mathbf{E}[X]$.
- Note the condition that $X$ must be a nonnegative random variable.

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- Answer: when $X < a$, $X$ should be 0; when $X \geq a$, $X$ should be $a$.
- The distribution for which Markov inequality tight is a two-point distribution.
- With this intuition, it is not difficult to prove the following corollary:

**Corollary (Reverse Markov Inequality)**

*If $X$ is a random variable that is never larger than $a$, then for any $b < a$,*

$$\mathbf{Pr}\left[X \leq b\right] \leq \frac{a - \mathbf{E}[X]}{a - b}.$$

# Application: Perfect Hashing

**Definition**

A hash function $h : U \to \{0, \ldots, m-1\}$ is *perfect* on $S \subseteq U$ if $\textsc{Find}(x)$ for every $x \in S$ takes $O(1)$ time.

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- It does not follow immediately that there exists an $h \in H$ under which every element has only $O(1)$ collisions.
  - In fact, with an "ideal hash", i.e., that sends every element in $U$ uniformly at random to $\{0, \ldots, m-1\}$, for $m = n$, w.h.p. the worst bucket has $\Theta(\log n / \log \log n)$ collisions.

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**Claim**

Let $H$ be a universal hash family from $U$ to $\{0, \ldots, m\}$, then for any $S \subseteq U$ with $|S| = n \leq \sqrt{m}$, for a random $h$ from $H$, with probability at least $\frac{1}{2}$, there is no collision under $h$.

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**Proof.**

By definition of universal hashing, for every $x \neq y$ in $S$, $\mathbf{Pr}_{h \sim H}[h(x) = h(y)] \leq \frac{1}{m}$.
By the union bound, the probability that any collision happens is at most $\sum_{x \neq y \in S} \frac{1}{m} < \frac{n^2}{2} \cdot \frac{1}{m} \leq \frac{1}{2}$. □

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    - For each $i = 0, \ldots, n-1$, let $n_i$ be the number of collisions in that bucket. Set up a hash table $B_i$ of size $n_i^2$, and a *perfect* hash function mapping $U$ to $\{0, \ldots, n_i^2 - 1\}$.

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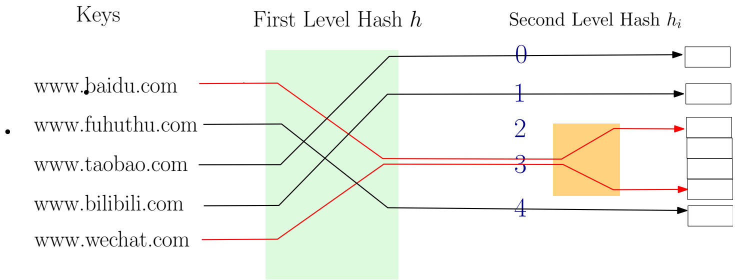
  - When looking up $x$, we first find its position in the first level. Let $j$ be $h(x)$. Then we look up $B_j[h_j(x)]$.

# Illustration: Perfect Hashing



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**Lemma**

*Let $h$ be sampled uniformly at random from a universal hash function family mapping $U$ to $\{0, \ldots, n-1\}$. Let $n_i$ be $|h^{-1}(i)|$, the number of elements mapped to $i$ by $h$. Then $\mathbf{Pr}[\sum_i n_i^2 \leq 4n] \geq \frac{1}{2}$.*

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**Proof.**

Game plan: we first show that $\mathbf{E}[\sum_i n_i^2]$ is no more than $2n$. Then the conclusion follows from Markov inequality.

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For $x \neq y$ in $S$, let $C_{xy}$ be the indicator variable for the event that $x$ clashes with $y$ under $h$, then $\mathbf{E}[C_{xy}] \leq \frac{1}{n}$ by universality.

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Now we can bound

$$\mathbf{E}\left[\sum_{x \in S} \sum_{y \in S \setminus \{x\}}\right] C_{xy} \leq n(n-1) \cdot \frac{1}{n} \leq n.$$

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Therefore $\mathbf{E}[\sum_i n_i^2] \leq 2n$. □

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- After $k$ trials, we succeed with probability $1 - \frac{1}{2^k}$.