Learning Goals

- Understand the design idea of skip lists
- Carry out more involved probabilistic runtime analysis using Chernoff bound and union bound
- Understand the idea of SkipNet in Peer-to-Peer systems

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  - One level above, at L_2, we have a linked list storing every four node from L_0, or every other node from L_1, also sorted, with $\lfloor n/4 \rfloor$ nodes, etc..
- Each copy of a node v in L_i stores pointers to v's copies in L_{i-1} and L_{i+1} (if they exist), and also the predecessor and successor in L_i.

Skip List: Illustration

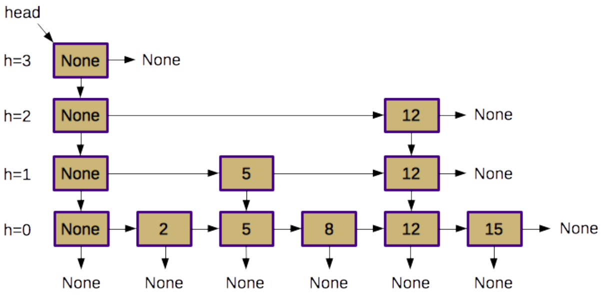

Image credit: Mike Lam at James Madison University

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- We may keep only the keys in levels other than L_0, and store the actual content only in nodes of L_0.
- Problem: INSERT and DELETE take time $O(n)$ if we were to keep the structures.

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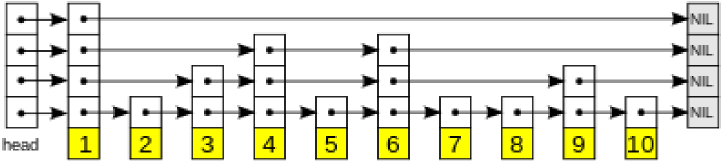
- The expected number of copies we insert for each node is 2.
- We need to show that this randomized construction yields similar performance for Find as the previous deterministic structure.

Randomized Skip List: ILlustration



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Analysis of FIND on Skip List

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- FIND may take too long for two possible reasons: there may be too many layers, or FIND takes too many horizontal steps.
- Let's first bound the number of levels H.
- The probability that a particular node has a copy at a level at least as high as H is 2^{-H}.
- By the union bound, when $n2^{-H} \leq \frac{1}{n^2}$, i.e., $H \geq 3 \log n$, with probability no more than $\frac{1}{n^2}$, there are no more than H levels.

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  - Otherwise, we step left.
- Once we reach level H, we declare success.
- The problem becomes: what's the probability that, after taking at least X steps, we haven't made H upward steps?

Apply Chernoff Bound

Take X to be, say, $36 \log n$, and let Y_i, $i = 1, \cdots, X$, be the indicator variable that the i-th step is upward. Then $\mathbf{E}[Y_i] = \frac{1}{2}$. Let Y be $\sum_i Y_i$.

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By Chernoff bound,

$$\begin{aligned}\mathbf{Pr}\left[Y \leq 3 \log n\right] &= \mathbf{Pr}\left[Y \leq \mathbf{E}\left[Y\right] - 15 \log n\right] \\ &\leq \exp\left(-\frac{2 \cdot (15 \log n)^2}{36 \log n}\right) < \frac{1}{n^2}.\end{aligned}$$

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This analysis was performed for a specific node x. By the union bound, with probability at least $1 - \frac{1}{n}$, no node takes more than $36 \log n$ steps to reach level H.

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- Now by a final union bound, with probability at least $1 - \frac{2}{n}$, there are no nodes beyond level $L_{3 \log n}$ and every node reaches that level within $36 \log n$ steps.
- So FIND takes time $O(\log n)$ for every node with high probability.

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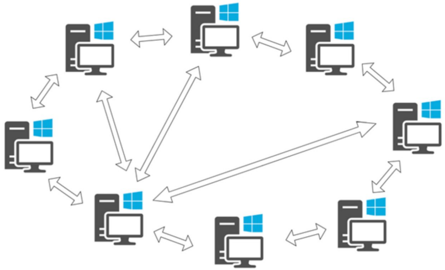

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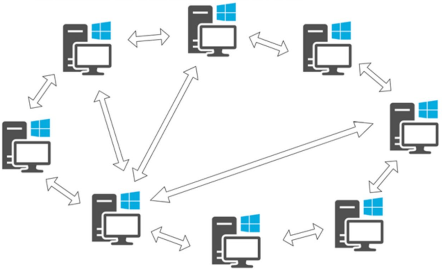

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- A request of a node to communicate with another can take $O(n)$ time to traverse the network if we are not careful.

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- To access a node, we go as far as possible on a high level, then descend and continue.