

Sparse Recovery

Application of Count-Sketch

Hu Fu @SHUFE, Oct 14, 2023

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Quantifying the error using $E_2^k(\mathbf{x})$ is necessary. It can be as large as comparable to $\|\mathbf{x}\|_2$

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Note the dependence on k

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Lemma. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, if $\|\mathbf{x} - \mathbf{y}\|_\infty \leq \frac{\epsilon}{\sqrt{k}} E_2^k(\mathbf{x})$, let \mathbf{z} be the k -sparse recovery of \mathbf{y} , then

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$$\text{Var}(z_i) = \mathbb{E}[(z_i - x_j)^2] = \mathbb{E} \left[\left(\sum_{j' \neq j} g_i(j)g_i(j')Y_{j,j'}x_{j'} \right)^2 \right] = \sum_{j' \neq j} x_{j'}^2 \mathbb{E}[Y_{j,j'}^2] \leq \frac{\|x\|^2}{w}$$

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 - With w growing linearly with k , this can be made to happen with small probability.

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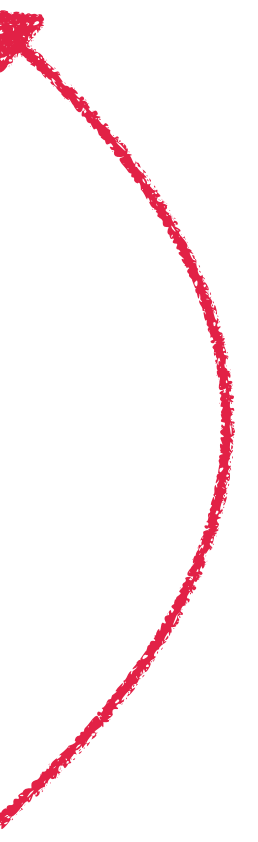
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By Chebyshev inequality, $\mathbb{P}[|z'_i - x_j| > \frac{\epsilon}{\sqrt{k}} E_2^k(\mathbf{x})] \leq \frac{1}{3}$.

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Let A be the event that none of entries in T collide with j under h_i , then $\mathbb{P}[A] \geq 1 - \frac{\epsilon^2}{3}$

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Let A be the event that some entry in T collides with j under h_i , then $\mathbb{P}[A] \leq \frac{k}{w} = \frac{\epsilon^2}{3}$

$$\begin{aligned} \mathbb{P}[|z_i - x_j| \geq \frac{\epsilon}{\sqrt{k}} E_2^k(\mathbf{x})] &\leq \mathbb{P}[A] + \mathbb{P}[|z'_i - x_i| > \frac{\epsilon}{\sqrt{k}} E_2^k(\mathbf{x}) \mid \bar{A}] \cdot \mathbb{P}[\bar{A}] \\ &\leq \frac{\epsilon^2}{3} + \frac{1}{3} \leq \frac{2}{5} \end{aligned}$$

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Recall the proof we gave for the performance of SkipList. We had a similar use of union bound.

Proof of Second Lemma

Lemma. If for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we have $\|\mathbf{x} - \mathbf{y}\|_\infty \leq \frac{\epsilon}{\sqrt{k}} E_2^k(\mathbf{x})$, let \mathbf{z} be the k -sparse recovery of \mathbf{y} , then

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• entries in $T \cap T'$ and $T' \setminus T$: by assumption, each entry contributes $\leq \frac{\epsilon^2}{k} (E_2^k(\mathbf{x}))^2$, and there are k of them

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By Cauchy-Schwartz, $\sum_{j \in T \setminus T'} |x_j| \leq \sum_{j \notin T} |x_j| \leq \sqrt{k \sum_{j \notin T} x_j^2} = \sqrt{k} E_2^k(\mathbf{x})$

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Putting everything together, $\|\mathbf{x} - \mathbf{z}\|_2^2 \leq (1 + 9\epsilon) (E_2^k(\mathbf{x}))^2$, hence $\|\mathbf{x} - \mathbf{z}\| \leq \sqrt{1 + 9\epsilon} E_2^k(\mathbf{x}) \leq (1 + 5\epsilon) E_2^k(\mathbf{x})$.

Putting Things Together.

Lemma. Count-Sketch with $w = 3k/\epsilon^2$, $\ell = O(\log n)$ guarantees $|x_j - \tilde{x}_j| \leq \frac{\epsilon}{\sqrt{k}} E_2^k$ for each j w.h.p.

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We can do faster by maintaining a record as the input comes!