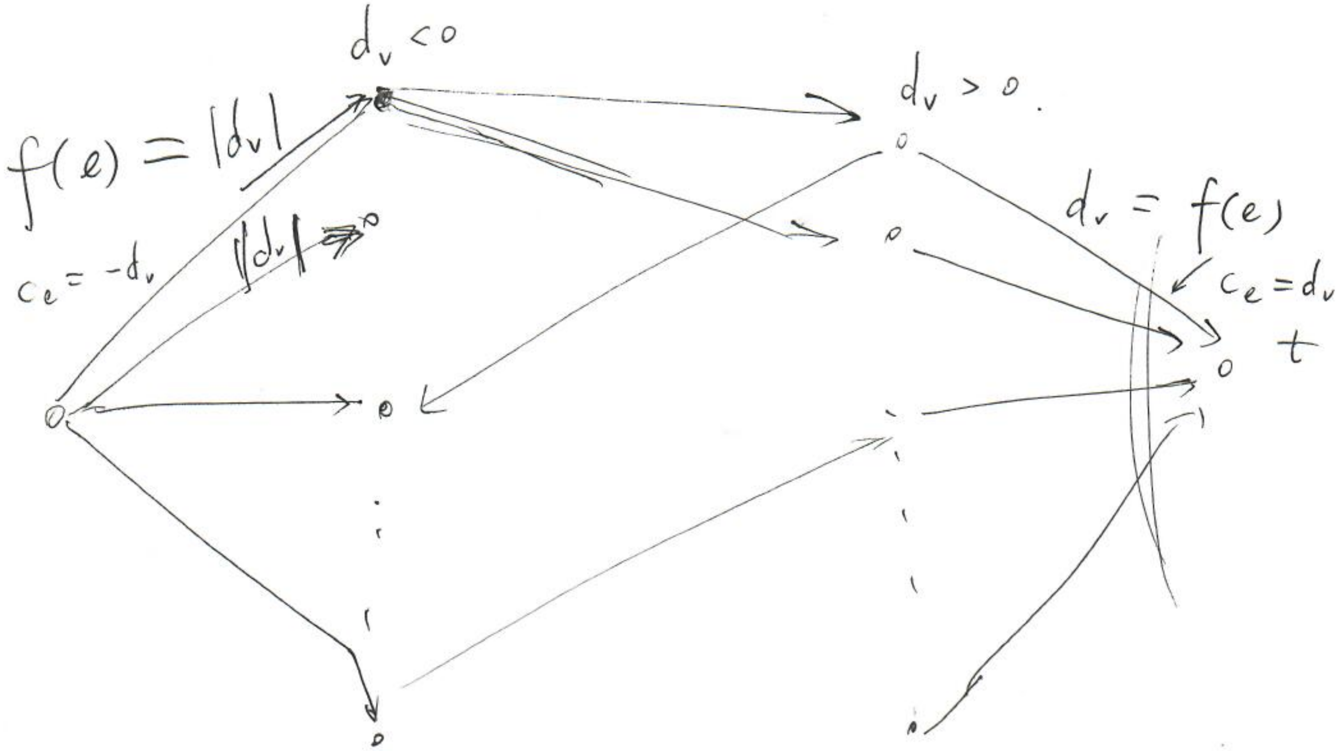


Conservation in flow (for a sender):

$$\underline{\underline{|d_v|}} + \sum_{\substack{e \in E \\ e \text{ into } v}} f(e) - \sum_{\substack{e \in E \\ e \text{ out of } v}} f(e) = 0.$$

$$\sum_{\substack{e \text{ into } v \\ e \in E}} f(e) - \sum_{\substack{e \text{ out of } v \\ e \in E}} f(e) = -|d_v| = d_v$$

Pf (Observation) $\sum d_v = 0$. $\Leftarrow$ Circulation exists

$\forall v \in V, \quad d_v = \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e)$

where $f$ is a circulation.

sum over $v$, $\sum_v \left[ \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) \right] = \sum_{v \in V} d_v$

any edge $(u,v)$ goes into $v$, out of $u$.

$LHS = \sum_e \left[ f(e) - f(e) \right] = 0$.

Pf (Correctness)

max flow that saturates the new edges

$\Rightarrow$ Circulation.

Circulation $\Rightarrow$ max flow saturating the new edges

Pf. Circulation w/o lower bound

$\iff$ Circulation w/ lower bound.

& $\Leftarrow$: Given circulation $f$ for the original graph, let $f'(e) = f(e) - l_e$.

$$\Rightarrow 0 \leq f'(e) \leq c_e - l_e. \quad (l_e \leq f(e) \leq c_e)$$

$$\forall v, \sum_{e \text{ into } v} f'(e) - \sum_{e \text{ out of } v} f'(e)$$

$$= \left( \sum_{e \text{ into } v} f(e) - \left( \sum_{e \text{ into } v} l_e \right) \right) - \sum_{e \text{ out } v} \left( f(e) - l_e \right)$$

$$\cancel{= \sum_{e \text{ out of}}} \qquad \cancel{= d'_v}$$

$$= \underline{\left[ \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) \right]} - \sum_{e \text{ into } v} l_e + \sum_{e \text{ out of } v} l_e$$

$$= d_v - \sum + \sum = d'_v$$