Application of Max Flow 2: Directed Edge-Disjoint Paths

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Running time: Ford-Fulkerson takes time $O(mn)$.

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A flow is said to have no cycle if, for any cycle $e_1, \ldots, e_k$ in the network, there is an e_j $(1 \leq j \leq n)$ such that $f(e_j) = 0$.

Lemma

In any flow network, there is a max flow with no cycle.

Extension: Undirected Edge-Disjoint Paths

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Solution: Show that one can remove such cycles from a flow without affecting the value of the flow.

Application of Max Flow 3: Survey Design

Problem: We have a set U of customers and a set V of products; each customer $u \in U$ has sampled a set $S_u \subseteq V$ of products. Is it possible to conduct a survey so that each customer u responds on no more than k_u products in S_u and each product is surveyed at least once?

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Reduce to a flow problem, similar to bipartite matchings:

- Set up a flow network, with source s, sink t, an edge from s to each node in U, an edge from each node in V to t, and an edge between (u, v) if $v \in S_u$; all edges have capacity 1, except edge (s, u) having capacity k_u;

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But what if we'd like to require more than one survey on each product? Say for product $v \in V$ we need at least L_v and at most M_v surveys?

(Added slide thanks to the discussion in class)

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- Solution 1: set the capacity on each edge (v, t) to be L_v, and observe that M_v is of no use.
- Solution 2: for each product v, set up L_v nodes, each having an edge going to t with capacity 1, and each having incoming edge (u, v) for every u such that $v \in S_u$.

What if we'd like to set up lower bounds as well for the customers? Say for product $v \in V$ we need at least L_v and at most M_v surveys, and customer $u \in U$ should do at least L_u and at most M_u surveys.

Extension of flow networks: Circulation with demands

- Input: a directed graph $G = (V, E)$; each node $v \in V$ has a *demand* $d_v \in \mathbb{R}$; each edge $e \in E$ has a capacity $c_e \geq 0$.

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- Output: a *circulation* if it exists. A circulation is a mapping $f : E \rightarrow R_+$ satisfying
  - (capacity condition) $\forall e \in E$, $0 \leq f(e) \leq c_e$;
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- Solution: Reduce to a flow problem!
  - Construct flow network G': add a source s and a sink t; for each $v \in V$ with $d_v < 0$, add an edge (s, v) with capacity d_v; for each $v \in V$ with $d_v > 0$, add an edge (v, t) with capacity $|d_v|$.

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Claim

A circulation exists in G if and only if a flow with value $\frac{1}{2}\sum_v |d_v|$ exists in G'. (Note: $\frac{1}{2}\sum_v |d_v|$ is just $\sum_{v:d_v>0} d_v$.)

Extension: circulation with demands and lower bounds

- Input: a directed graph $G = (V, E)$; each node $v \in V$ has a *demand* $d_v \in \mathbb{R}$; each edge $e \in E$ has a *lower bound* $\ell_e \geq 0$ and a capacity $c_e \geq \ell_e$.

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  - Let d'_v be $d_v - \sum_{e \text{ into } v} \ell_e + \sum_{e \text{ out of } v} \ell_e$; let c'_e be $c_e - \ell_e$.

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Exercise: Reduce the survey design problem to circulation with demands and lower bounds (or read the reduction in textbook).