Learning Goals

- (Reviewing) basics of probabilities: events, independence, union bound.
- Contention resolution with random access, and analysis of its efficiency
- Some facts about repeated tosses of a biased random coin

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- Trivial if the tasks can agree on some ordering and requests the service one by one.
- Problem: The tasks cannot talk with each other and there is no central authority.
- **Randomized strategy:** In each time step, each task requests with some small probability p, independently.

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- Let $S[i,t]$ denote the event that task i sends a request at time t *and* gets served, then

$$\mathbf{Pr}\left[S[i,t]\right] = \mathbf{Pr}\left[A[i,t] \cap \bigcap_{j \neq i} \overline{A[j,t]}\right] = p(1-p)^{n-1}.$$

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- To maximize $\mathbf{Pr}[S[i,t]]$, set $p = 1/n$.

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We set p to maximize $\mathbf{Pr}[S[i,t]]$ to $\frac{1}{n}(1-\frac{1}{n})^{n-1}$. How good is this?

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So $1/(en) \leq \mathbf{Pr}[S[i,t]] \leq 1/(2n)$. Therefore $\mathbf{Pr}[S[i,t]]$ is asymtotically $\Theta(1/n)$.

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  - Remark: often, in many situations, the two give answers that are close: sometimes one may show that the random quantity *concentrates* around its expectation. *Tail bounds* are used to prove this.
- Probability with which task i does not succeed in the first t steps:

$$\mathbf{Pr}\left[\cap_{r=1}^{t}\overline{S[i,r]}\right] = \prod_{r=1}^{t}[1-\mathbf{Pr}\left[S[i,r]\right]] = \left[1-\frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}\right]^{t}.$$

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- Big picture (very useful high level intuition): if we have a biased coin that gives Heads with probability $1/k$:
  - In about k independent tosses, one "expects" to see a Heads;
  - However, with constant probability, a Heads doesn't show in k tosses;
  - But if one tosses the coin $\theta(k \log k)$ times, the probability that no Heads shows up quickly tends to 0.

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- Let $F[i,t]$ denote the event that task i fails in the first t steps, we have shown $\mathbf{Pr}[F[i,t]] \leq e^{-t/en} = n^{-c}$ for $t = \lceil en \cdot c \ln n \rceil$.

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$$\mathbf{Pr}\left[\cup_{i=1}^{n} F[i,t]\right] \le \sum_{i=1}^{n} e^{-t/en} = ne^{-\frac{t}{en}}.$$

So for $t = \lceil 2en \ln n \rceil$, this is at most $\frac{1}{n}$.