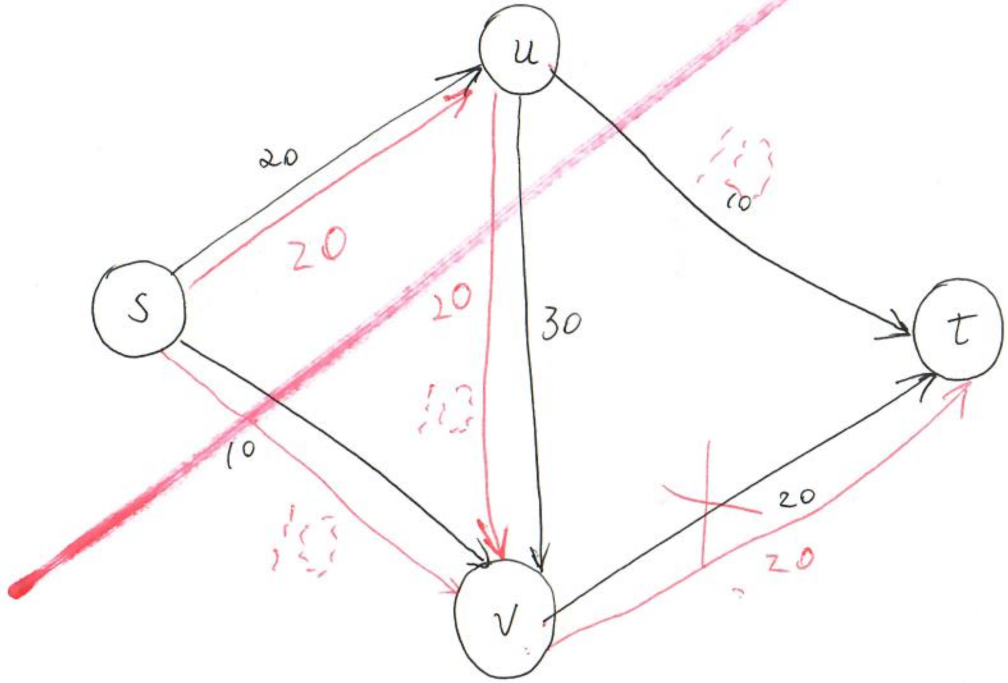
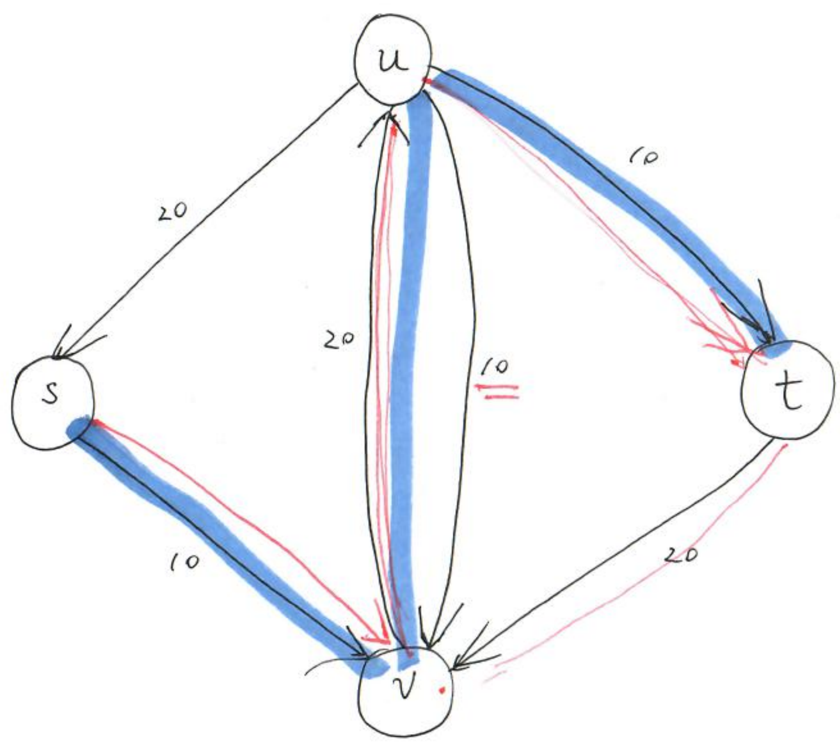
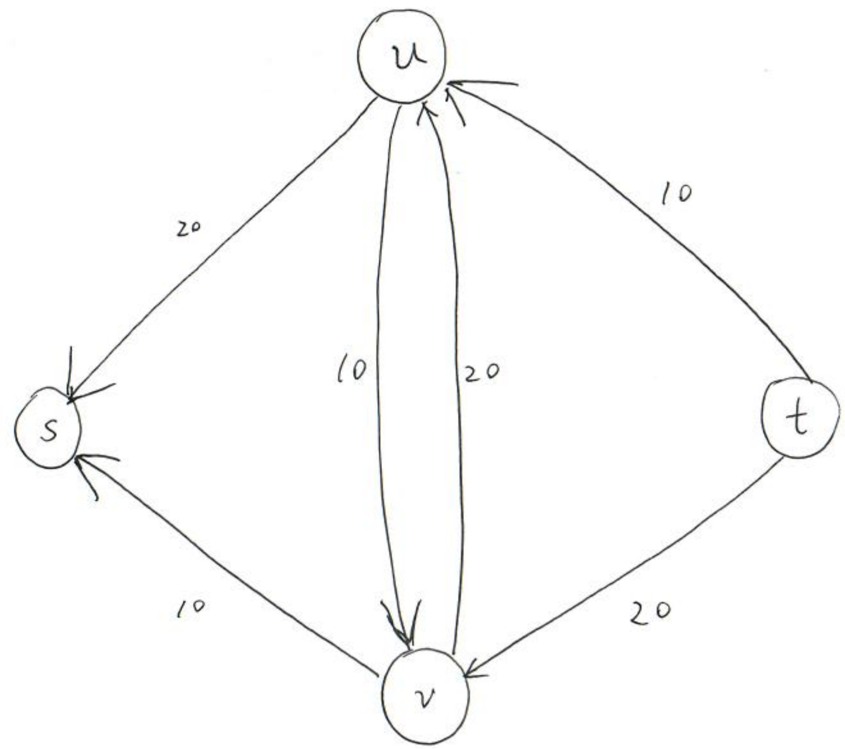


Flow Network Example



Residual Graph G_f



Residual Graph after another Augmentation

Pf (f' is flow)

Capacity: $\forall e \in G$, if $f(e) = f'(e)$. ✓

if $f(e) \neq f'(e)$: either e is a forward edge in G_f,

$$f'(e) = f(e) + b$$, where b is bottleneck of the augmenting path.

$$b \leq c_e - f(e) \leftarrow \text{residual cap. of } e.$$

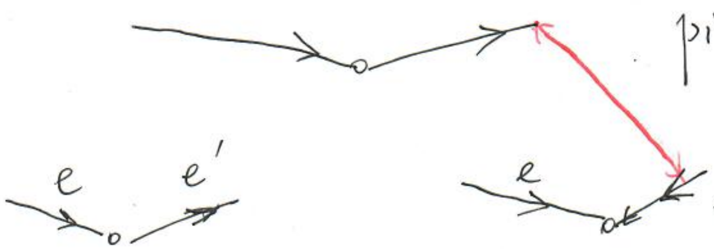
$$0 \leq f'(e) \leq f(e) + c_e - f(e) \leq c_e.$$

$e = (u,v)$, but the path in G_f uses (v,u), which is a backward edge, $f'(e) = f(e) - b \geq 0$.

$$c_e \geq f(e) \geq f'(e) \geq 0.$$

$$b \leq f(e) \leftarrow \text{residual capacity on } (v,u).$$

Conservation: $\forall v \notin \{s,t\}$,



picture in G_f

pic in G

$f'(e) = f(e) + b$
$f'(e') = f(e') + b$

$f'(e) = f(e) + b$
$f'(e') = f(e) - b$

Lemma $\forall f$ flow, $\forall$ s-t cut (A, B)

$$f^{out}(A) - f^{in}(A) = |f| = v(f)$$

Pf. Induct on $|A|$.

When $|A| = 1$, $A = \{s\}$. $|f| = f^{out}(A) - f^{in}(A)$

IH: Suppose this is true for any A, $|A| \leq k$.

Consider A, $|A| = k+1$.

Take $v \neq s$ in A,

$$f^{out}(A) = f^{out}(A \setminus \{v\}) + \sum_{\substack{e \text{ out of } v \\ \text{to } B}} f(e) - \sum_{\substack{e \text{ out} \\ \text{of } A\setminus\{v\} \text{ into } v}} f(e)$$

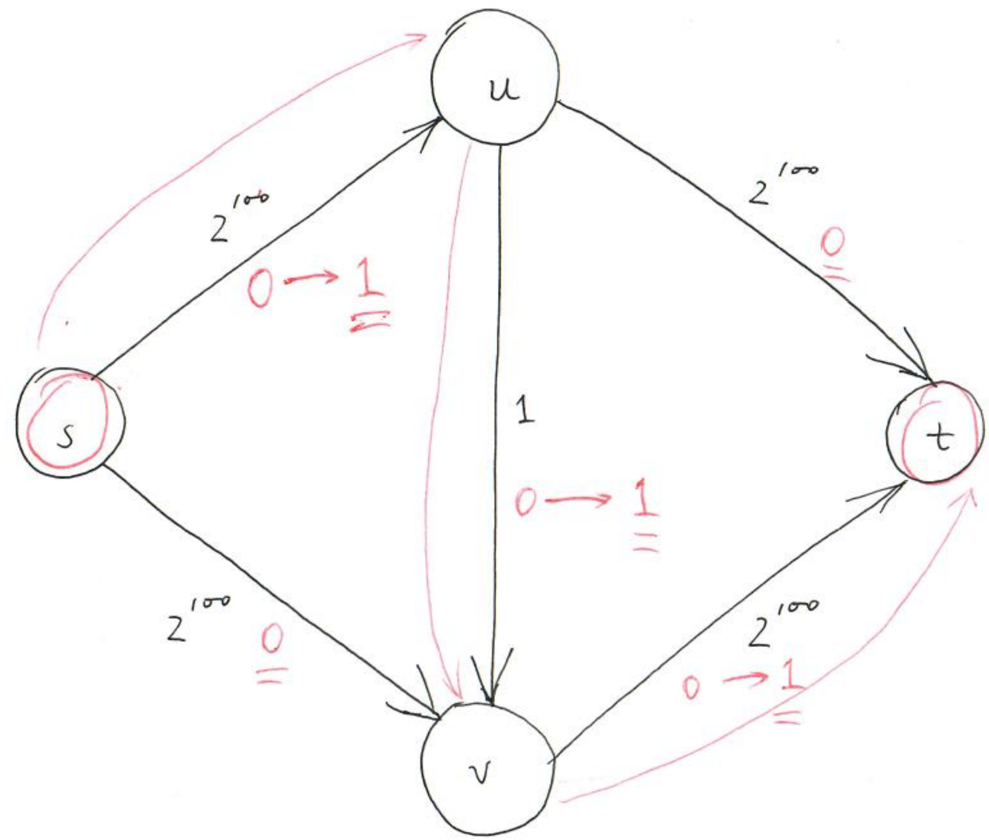
$$f^{in}(A) = f^{in}(A \setminus \{v\}) + \sum_{e \text{ into } v \text{ from } B} f(e) - \sum_{e \text{ from } v \text{ to } A\setminus\{v\}} f(e)$$

$$\Rightarrow \quad f^{out}(A) - f^{in}(A) = f^{out}(A\setminus\{v\}) - f^{in}(A\setminus\{v\})$$

$$+ \underbrace{\left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e)\right)}_{=0}$$

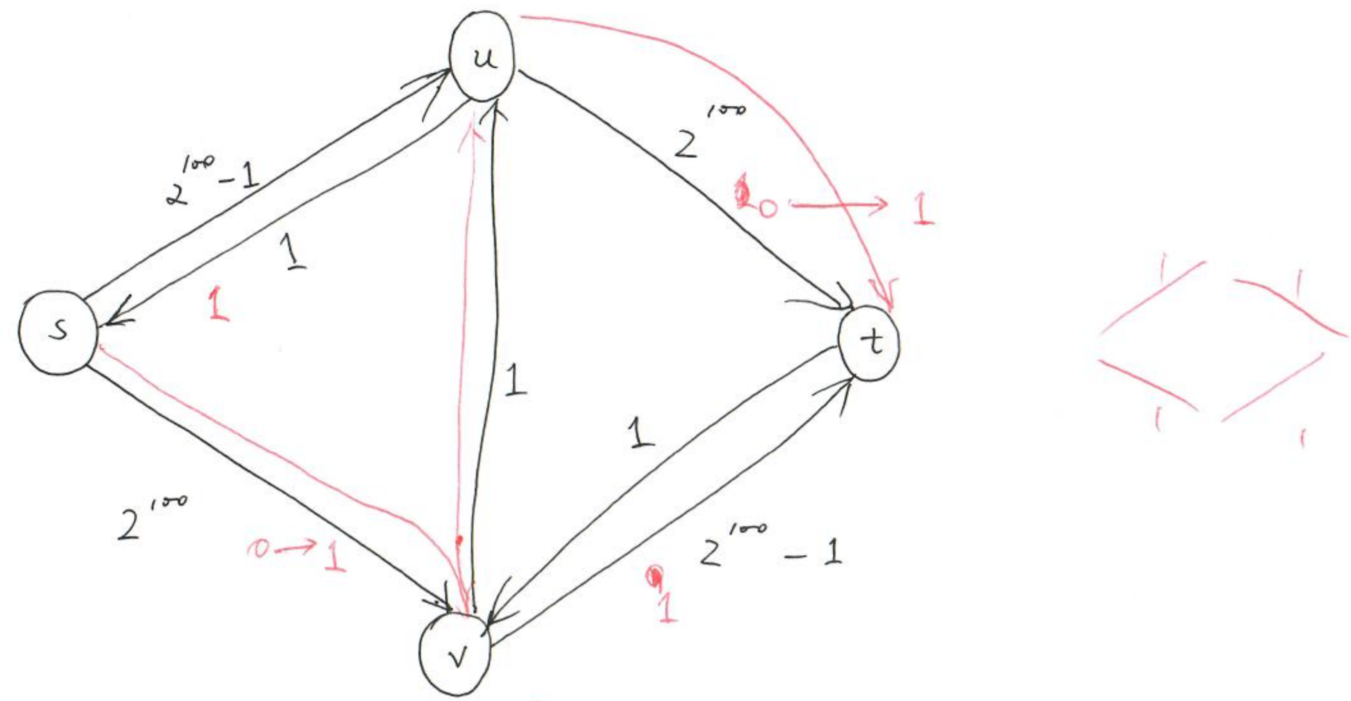
conservation at v.

$$= f^{out}(A\setminus\{v\}) - f^{in}(A\setminus\{v\}) = |f|.$$

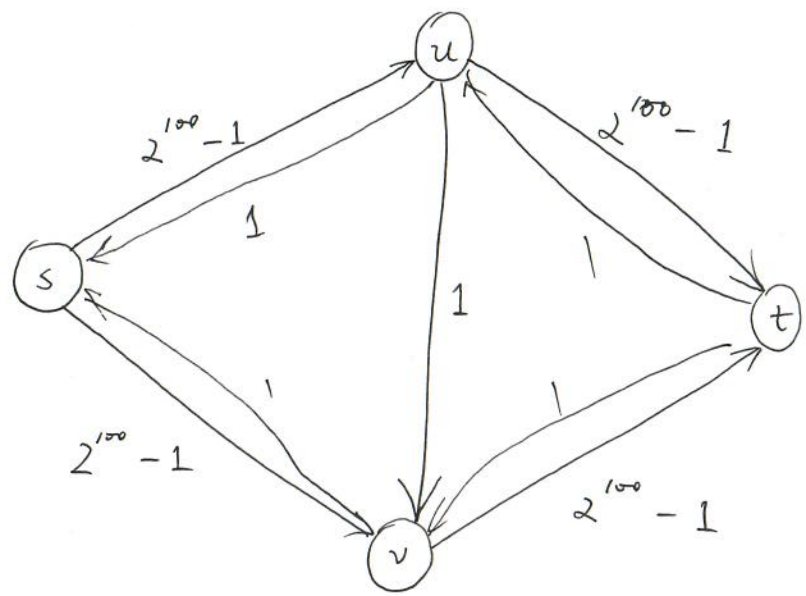


Bad example for Ford-Fulkerson Alg.

$f(u,v)=1 \longrightarrow f(u,v)=0$.



Residual graph after one augmentation



Residual graph after 2 augmentations

Pf. (Corollary) $\forall e$ out of A, $f(e) \le c_e$

$$\Rightarrow f^{out}(A) \le c(A, B)$$

By Lemma, $|f| = f^{out}(A) - f^{in}(A) \le c(A,B) \underline{\underline{- f^{in}(A)}}$

$$\le c(A, B)$$

Pf (Max Flow Min Cut)

there exists

① ⇒ ③ : Proof by contradiction. If $\exists$ augmenting path, then augmenting along increases the value of the flow by the bottleneck (>0). $\Rightarrow\Leftarrow$

② ⇒ ① : By ~~coll~~ corollary, $\forall f'$, $|f'| \le c(A, B) = |f|$.

③ ⇒ ② : Let A be the set of nodes reachable from s in the residual graph G_f, and $B = V \setminus A$.

Consider any edge $(u, v) \in E$, $u \in A$, $v \in B$, (u, v) is not in G_f. $\Rightarrow f(e) = c_e$.

$f^{out}(A) = c(A, B)$. $\forall (u, v) \in E$, $u \in B$, $v \in A$.

(v, u) is not in G_f $\Rightarrow f((u, v)) = 0$. $\Rightarrow f^{in}(A) = 0$.

$\Rightarrow |f| = f^{out}(A) - f^{in}(A) = c(A, B)$