Learning Goals

- A simple randomized algorithm computing min cuts
- Las Vegas and Monte Carlo algorithms
- Conditional probabilities and their use in the analysis of randomized algorithms
- Improving the precision of a randomized algorithm by repetition

The Min Cut Problem

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- Given an undirected graph $G = (V, E)$, for $S \subsetneq V$, $S \neq \emptyset$, let $c(S)$ be $|\{(u, v) \in E : u \in S, v \notin S\}|$, called the volume of the cut $(S, \overline{S})$.

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- Question: find S that minimizes $c(S)$.
- Deterministic algorithm: fix $s \in V$, then for each $t \neq s$, find the minimum number of edge-disjoint paths from s to t, corresponding to an "s-t" cut with a volume equal to the number of such paths. Take the one with the smallest volume.

A simple randomized algorithm due to David Karger

- choose an edge (u, v) uniformly at random and *contract* it: merge u and v into a new node w, which inherits all other edges incident to u or v.
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Theorem

The algorithm finds a minimum cut with probability at least $1/\binom{n}{2}$, *where* $n = |V|$.

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- How do we use Monte Carlo algorithms?
- Repeat many times independently, and take the best solution: with high probability it is optimal.
- Example: for the min cut algorithm, repeat $n^2 \log n$ times, the probability that no correct solution shows up is $\leq 1/n$.

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- ~~The minimum degree never decreases after an edge is contracted.~~ As long as no edge in C is contracted, C remains a min cut.

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- Example 4: Total probability rule:

$$\mathbf{Pr}[A] = \mathbf{Pr}[A \mid B]\,\mathbf{Pr}[B] + \mathbf{Pr}\left[A \mid \overline{B}\right]\mathbf{Pr}\left[\overline{B}\right].$$

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  - Answer: 1/3. A less deceptive version: Conditioning on that not both of them are girls, what is the probabilty that both are boys?

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  - Let the atom events be BB, BG, GB, GG, each with probability 1/4. Then event A, "one of them is a boy", is {BB, BG, GB}, and event B, "both are boys", is {BB}. So $\mathbf{Pr}[B \mid A] = \frac{|A \cap B|}{|A|} = 1/3$.

Using Conditional Probabilities

- Key observation: conditioning on $\mathcal{E}_1, \cdots, \mathcal{E}_i$, $\mathbf{Pr}[\mathcal{E}_{i+1} \mid \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_i] \geq 1 - \frac{2}{n-i}$.

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- Therefore,

$$\begin{aligned}
\mathbf{Pr}\left[\cap_{i=1}^{n-2}\mathcal{E}_i\right] &= \mathbf{Pr}\left[\mathcal{E}_1\right] \cdot \mathbf{Pr}\left[\mathcal{E}_2 \mid \mathcal{E}_1\right] \cdots \mathbf{Pr}\left[\mathcal{E}_{n-2} \mid \cap_{i=1}^{n-3}\mathcal{E}_i\right] \\
&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\cdots\left(1 - \frac{2}{3}\right) \\
&= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{2}{4} \cdot \frac{1}{3} \\
&= \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}.
\end{aligned}$$