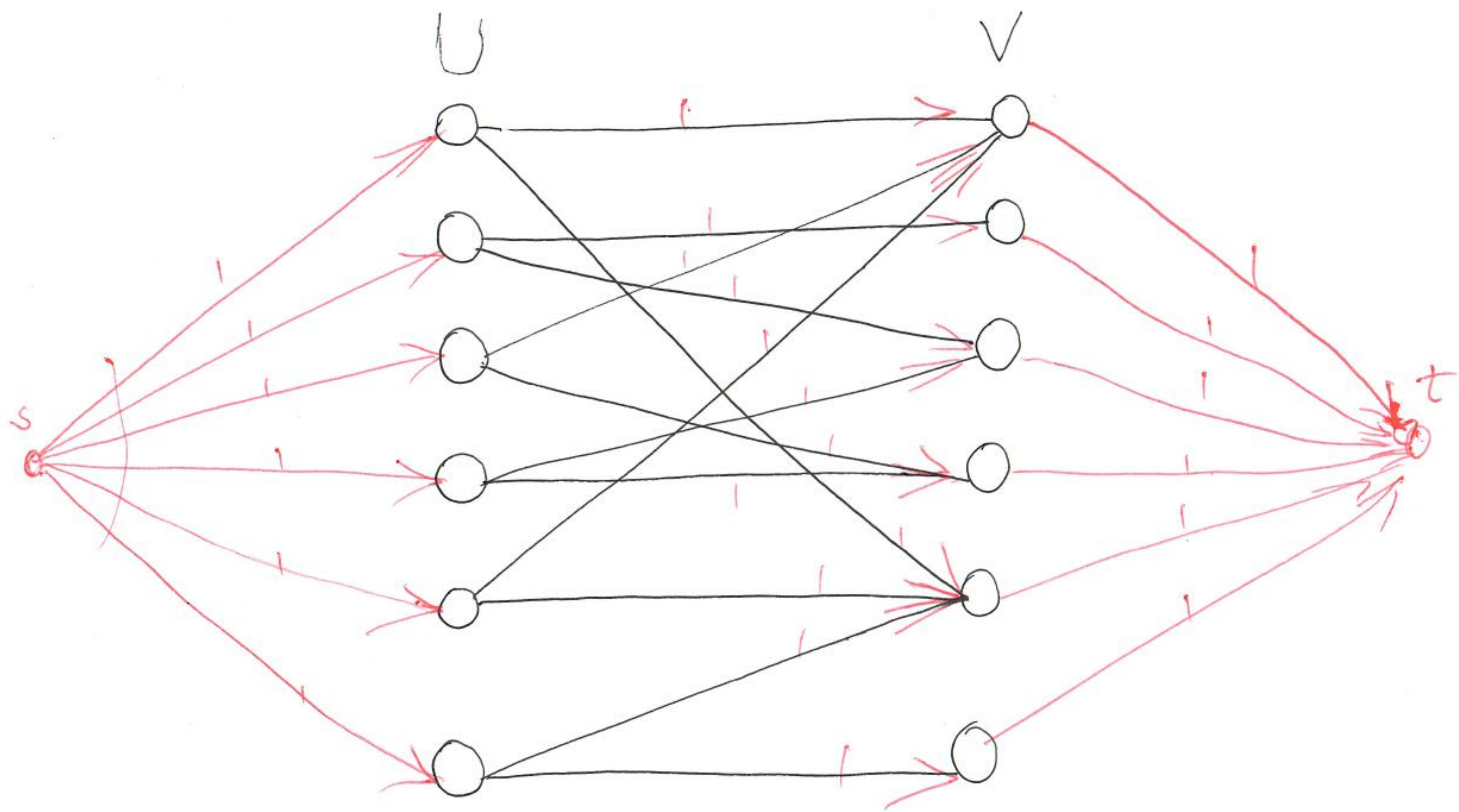

Bipartite Matching Example

One to one correspondence btwn matching and flow.

Matching M $\Rightarrow$ Flow: For each $e = (u, v) \in M$,

~~send~~ let $f(s, u) = f(u, v) = f(v, t) = 1$.

For any other edge e untouched in this procedure, $f(e) = 0$.

① $|f| = |M|$. ② f is a flow.

Pf ②: Obviously, any e ~~(u, v)~~, $0 \leq f(e) \leq 1$. (Capacity constraint is checked)

Conservation: ~~$\forall (s, u)$~~, $\forall u \in U$, if $f((s, u)) = 1$,

in which case $\exists (u, v) \in M$, $f(u, v) = 1$,

also, $\nexists v' \neq v$, s.t. $(u, v') \in M$. (because of matching)

hence ~~f~~ (s, u) is the only edge into u

thas carries positive flow, and (u, v) the

only edge out of u w/ pos. flow.

Same argument for $v \in V$.

Pf ①. Consider ~~cut $(\{s\} \cup \ldots, V \cup \{t\})$~~

$|f| = \sum_{u \in U} f((s, u))$,

$= \sum_{u \in U : \exists v, (u, v) \in M} 1 = |M|$.

each edge in M ~~has~~ is incident to exactly one node in U, and no two edges in M share a node in U.

Integral valued

② Flow $\Rightarrow$ Matching: $\forall (u,v)$, $u \in U$, $v \in V$, $f((u,v)) = 1$, ~~add~~ include (u,v) in M.

Show that ① M is a matching ② $|M| = |f|$.

Pf: ① $\forall u \in U$, the number of edges in M incident to u is at most 1, because otherwise $\exists\, v, v' \in V$ s.t. $\underbrace{f((u,v)) = f((u,v'))}_{=1}$ which means $f((s,u)) \geq 2$, violating the capacity constraint.

The same argument for any $v \in V$.

② From ①, we see that $\forall u$, the number of edges in M that are incident to u is equal to $f((s,u))$. Therefore

$$|M| = \sum_{u \in U:} |\{(u,v) \in M\}| = \sum_{u \in U} f((s,u)) = |f|.$$

~~Pf~~ (Hall's) If $\forall S \subseteq U$, $|\delta(S)| \geq |S|$, then perfect matching exists.

Pf. By the reduction, it suffices to show that ~~the~~ a max flow in the flow network has value n.

By max flow min cut thm, it suffices to show that ~~w~~the min. capacity of ~~an~~ cut is n.

Take any s-t cut, we show its capacity is $\geq n$.

Generically, the cut can be written as

$$\left(\{s\} \cup A \cup B,\ \{t\} \cup (U \setminus A) \cup (V \setminus B) \right)$$

for $A \subseteq U$, $B \subseteq V$.

The capacity of this cut is

$$\geq |U \setminus A| + |\delta(A) \setminus B| + |B|$$

$$\geq |U \setminus A| + |\delta(A)| \geq |U \setminus A| + |A| = n$$

$$|\delta(A) \setminus B| = |\delta(A)| - \underbrace{|\delta(A) \cap B|}_{\leq |B|}$$

$$|\delta(A) \setminus B| + |B| \geq |\delta(A)|$$

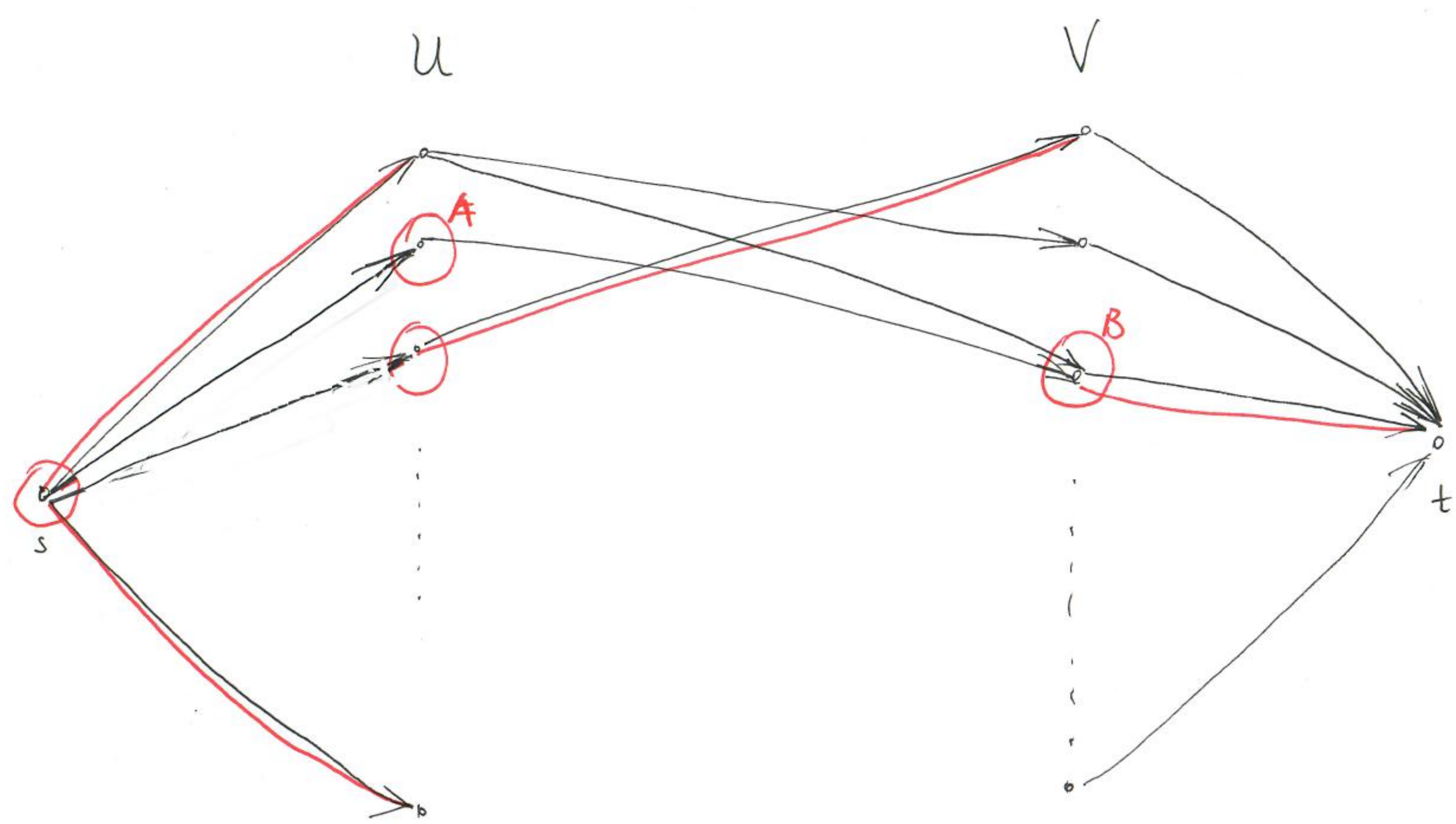

Illustration of Proof of Hall's Thm.

Nodes in red circles are on the side of s in the cut $\cancel{(\{s\} \cup A \cup B, \{t\}}$
$(\{s\} \cup A \cup B, \{t\} \cup (U \setminus A) \cup (V \setminus B))$

Edges highlighted in red contribute to the capacity of this cut.

From s to U, there are $|U \setminus A|$ edges "in the cut".
From V to t, there are $|B|$ edges "in the cut".

From U to V, there are $\geq |\delta(A) \setminus B|$ edges "in the cut".