Induction step: (for Dijkstra's Algorithm)

If IH is true for $|S| = k$, suppose $u$ is added to $S$, with $p(u) = v$. Need to show $P_u$ is the shortest path from $s$ to $u$. Consider any other path $P'$ from $s$ to $u$.

Case 1: If $P'$ goes only through nodes in $S$. Suppose the last node before $u$ is $w$. By IH, from $s$ to $w$ along $P'$ the cost is $\geq d(w)$. When $w$ was added to $S$, we updated $d(u)$ so that $d(u) \leq d(w) + c_{(w,u)}$.

Therefore $d(u) \leq d(w) + c_{(w,u)} \leq$ cost of $P'$.

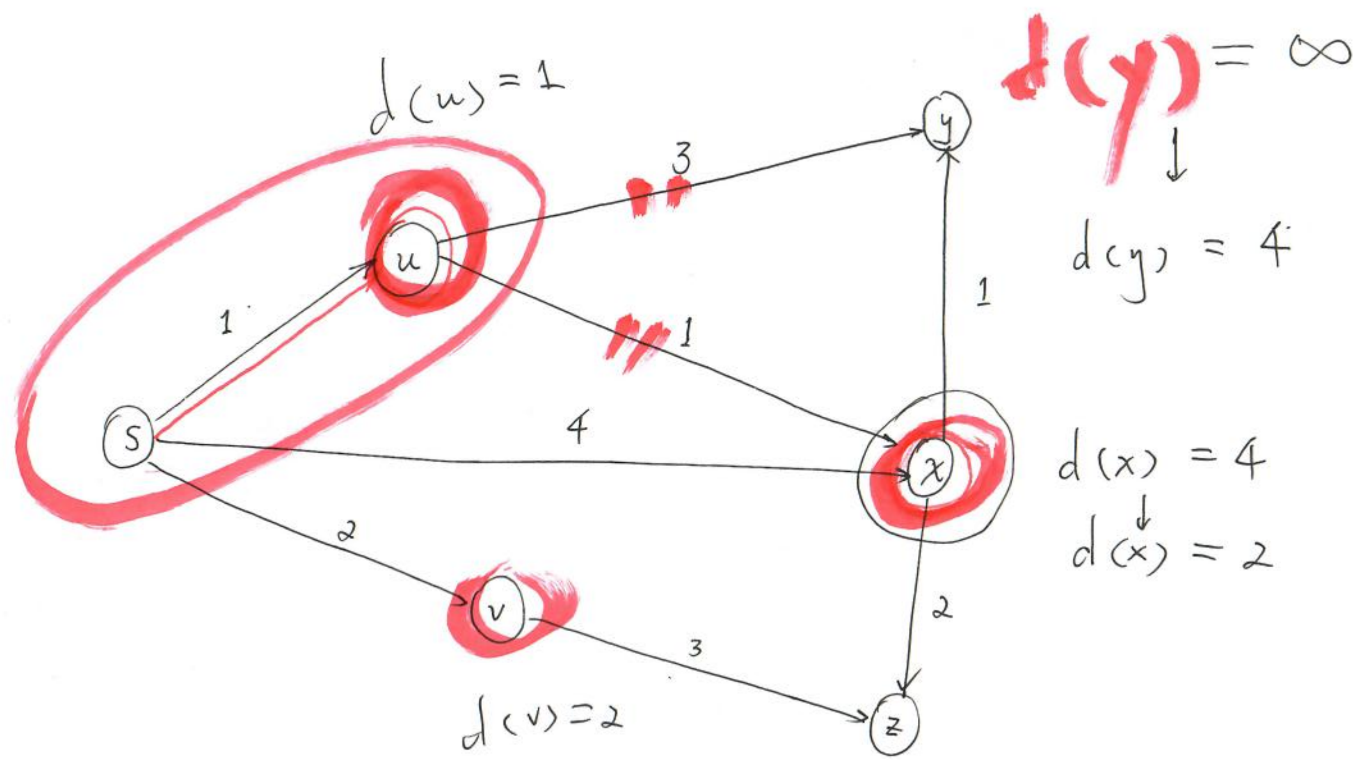
Case 2: If $P'$ leaves $S$ at some point. Suppose $P'$ leaves $S$ and reaches $v'$ from $w \in S$. Then ~~co~~ by nonnegativity of costs, cost of $P' \geq$ cost of $P'$ from $s$ to $v'$ ~~≥~~; by ~~case 1~~ IH, this is $\geq d(w) + c_{(w,v')}$.

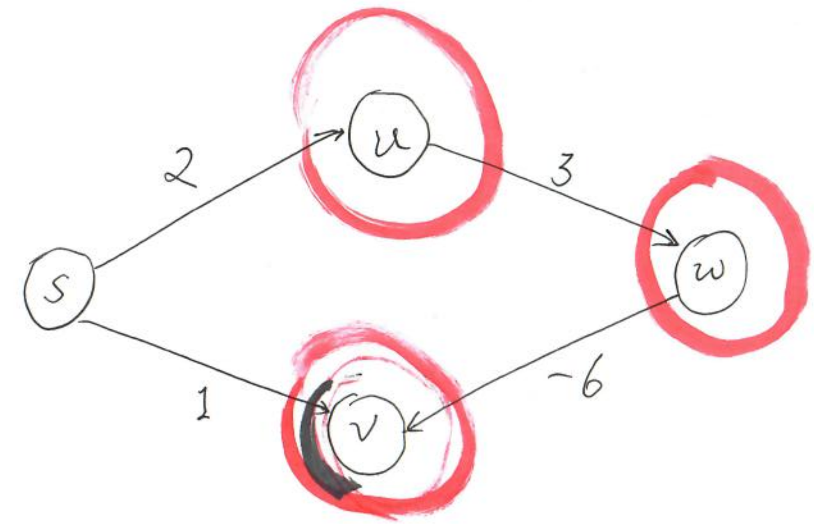
When $w$ was added to $S$, we made sure $d(w) + c_{(w,v')} \geq d(v')$.

Since $u$ was chosen to minimize $d$, we have $d(v') \geq d(u)$.

Altogether, we have cost of $P' \geq d(u)$.

Therefore $P_u$ is indeed the shortest path from $s$ to $u$.


d(u) = 1
d(v) = 2
d(y) = ∞
d(y) = 4
d(x) = 4
d(x) = 2
s
u
v
x
y
z
1
3
1
4
2
3
1
2


s
u
w
v
2
3
1
-6

Proof (Induction) of Bellman-Ford.

Base case, $i=0$, obvious.

Given the IH for $i \le k-1$, let's prove when $i = k$.

Take any path ~~of~~ ~~ten~~ with $\le k+1$ edges from $s$ to $v$. Let the last edge by $(w,v)$.

By IH, the part of path from $s$ to $w$ (having $\le k$ edges) has cost $\ge d_{i-1}(w)$.

Therefore the path costs $\ge d_{i-1}(w) + c_{(w,v)}$

But the alg. makes sure $d_i(v) \le d_{i-1}(w) + c_{(w,v)}$.

$d_i(w) \le d_{i-1}(w)$.