Learning Goals

- Random variables and their expectations
- Expectation of some distributions (Indicator variables/Bernoulli, binomial, geometric)
- Linearity of expectations
- Guessing cards and coupon collection

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- Example: For an event A, let X be 1 if A happens, and 0 if not. Then $\mathbf{Pr}[X = 1] = \mathbf{Pr}[A]$.
  - X is called the *indicator variable* of A.
  - A random variable that only takes values 0 or 1 is said to be drawn from a *Bernoulli distribution*.

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For $x < 1$,

$$\sum_{j=1}^{\infty} jx^{j-1} = \sum_{j=1}^{\infty} (x^j)' = \left(\sum_{j=1}^{\infty} x^j\right)' = \left(\frac{1}{1-x} - 1\right)' = \frac{1}{(1-x)^2}.$$

Linearity of expectation

- For random variables X and Y defined on the same probability space, a new random variable $X + Y$ is given by $(X + Y)(\omega) = X(\omega) + Y(\omega)$ for any sample point (that is, atom event) ω.

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For any collection of random variables $X_1, \cdots, X_n$ (defined on the same probability space),

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Remark: Linearity does NOT need indepedence between the variables!

Diversion: Independence among random variables

Definition

Two random variables are *independent* if for any i, j, the events $X = i$ and $Y = j$ are independent.

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Shuffle a deck of n distinct cards, and reveal them one by one. Before each revelation, make a uniformly random guess. How many guesses are correct in expectation?

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$$\mathbf{E}\left[X\right] = \sum_{i=1}^{n} \mathbf{E}\left[X_i\right] = \sum_{i=1}^{n} \frac{1}{n-i+1} = H(n) \approx \ln n.$$

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- $\mathbf{E}[X_i] = \frac{n}{n-i+1}$.
- Therefore the expected total number of purchases is

$$\sum_{i=1}^{n} \frac{n}{n-i+1} = n \cdot \sum_{i=1}^{n} \frac{1}{i} = nH(n) \approx n \ln n.$$