Setup and the algorithm

The MAX 3-SAT problem:

- Input: A 3-SAT instance, with n variables $x_1, \ldots, x_n$, and m clauses $C_1, \cdots, C_m$, each having three literals formed from distinct variables.
- Output: a truth assignment to the variables that satisfies as many clauses as possible.

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- Claim: In expectation, $\frac{7}{8}m$ clauses are satisfied.

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- Then the total number of satisfied clauses is $Y := \sum_i Y_i$, and $\mathbf{E}[Y_i] = \mathbf{Pr}[C_i \text{ satisfied}] = \frac{7}{8}$.
- By linearity of expectation,

$$\mathbf{E}[Y] = \mathbf{E}\left[\sum_i Y_i\right] = \sum_i \mathbf{E}[Y_i] = \frac{7}{8}m.$$

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Proposition

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Proof: "Expectation/average $\leq$ maximum".
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Remark

This is an example of the *probabilistic method*: showing the existence of an object by showing that it occurs with positive probability (often in a constructed probability space).