Job Security, Stability and Production Efficiency*

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Abstract

We study a two-sided matching market with a set of heterogeneous firms and workers in an environment where jobs are secured by regulation. Without job security Kelso and Crawford have shown that stable outcomes and efficiency prevail when all workers are gross substitutes to each firm. It turns out that by introducing job security, stability and efficiency may still prevail, and even for a significantly broader class of production functions.

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1 Introduction

Since the work of Kelso and Crawford (1982) the two-sided many-to-one matching model has emerged as the prominent tool to analyze labor markets whenever firms and workers are heterogeneous. The notion of stability, initially due to Gale and Shapley (1962), is the standard solution concept for matching models in general and for labor markets in particular. A stable outcome is an allocation of workers to firms (of which one firm is the outside option of unemployment) and a salary vector for the workers such that no combination of a single firm and a set of workers can improve their position while disregarding the others (there is no "blocking coalition"). Underlying the logic of this solution concept is the notion of a free, unregulated, competitive market, where any coalition can withdraw from the market if the market does not provide them with a desired outcome.

A fundamental question about stability, as with any game-theoretic (or economic) solution concept, is its existence. An elegant solution concept whose existence cannot be guaranteed in settings of economic interest falls short of being fully satisfactory. In their original paper, Kelso and Crawford prove existence, as well as efficiency, under the assumption that firms' preferences over sets of workers exhibit "gross substitutability" (on which we elaborate in the sequel). Much of the follow-up literature followed in their footsteps and assumes gross-substitutes production functions. In fact, Gul and Stacchetti (1999) have shown that existence of stable outcomes may not be guaranteed beyond gross substitutes production functions and the theory then becomes mute for such markets. To remedy this, we consider the following question: can one weaken the requirements underlying the notion of stability, in some natural way, to obtain existence for a larger class of markets?

In reality many labor markets are regulated and in particular much of the regulation provides various degrees of job security to workers.[1] The theoretical literature on matching seems to be mute about the possibility and implications of job security, and the ongoing public debate of such regulation has not been part of the matching literature so far. Job security regulation, within the context of a matching model, should be seen as a hurdle to the formation of blocking coalitions. Under such regulation, one should expect stability to hold for a larger class of production functions. This is exactly the line of thought we pursue.

Thus, partly to remedy the existence problem of stable outcomes and partly motivated by observations about real labor markets, the present paper studies matching markets that enforce

[1] In most European countries many employees have indefinite contracts which make it very difficult and very costly for an employer to terminate a contract. In the UK, for example, the tenure necessary to qualify for such protection was lowered in 1999 from 24 to 12 months (Marinescu, 2009). In Germany, the 1951 Dismisal Protection Act which is still largely valid today acknowledges that workers have the right to keep their jobs, and, for example, fixed term contracts are allowed only for a period of up to 18 months (Emmenegger and Marx, 2011). High job security exists in many non-European countries as well. In India, as another example, the Industrial Disputes Act of 1976 requires that written permission to retrench workers be obtained, normally from the relevant state government (Fallon and Lucas, 1991).

job security. Our contribution is conceptual as well as technical. Conceptually, we introduce a new solution concept for the many-to-one matching model: JS-stability. We do so by revising the notion of stability so it accounts for a regulated labor market. In particular we would like to model a regulated market where firms cannot unilaterally fire employees, or where such costs of firing are prohibitively high. In such labor markets, for a firm to be part of a blocking coalition, it must account for its current employees and ensure their utility is not compromised. More simply, such a firm must retain its workers at their current salary level. Technically, such regulation implies fewer blocking coalitions. Consequently, the requirements underlying the implied notion of stability, which we refer to as JS-stability (where JS stands for Job Security), become easier to satisfy.

It is no surprise, therefore, that we can guarantee the existence of JS-stable outcomes in some markets where no stable outcomes exist. As previously discussed, a key assumption for most results on labor markets is that of gross substitutability. In the Kelso and Crawford model that we adopt, such gross substitutability is a necessary and sufficient condition for a variety of results (see, among others, Kelso and Crawford (1982), Gul and Stacchetti (1999) and Ausubel (2006)). Our treatment, on the other hand, goes substantially beyond the scope of gross substitutability and allows for a broader class of preferences. Recall that a production function is called submodular if it exhibits decreasing productivity. It is well-known that the class of submodular production functions strictly contains the class of gross-substitutes production functions, and in fact significantly expands it. The classes of production functions that we study strictly contain and significantly expand the class of submodular production functions.

Our analysis starts by providing analogs of the welfare theorems to markets with job security, using these new concepts. On the one hand, we show that existence and optimality of a JS-stable outcome is guaranteed for a class of "almost fractionally subadditive" valuations (AFS), which we formally define in the sequel. On the other hand, although there may be inefficient JS-stable outcomes, we provide a tight bound on the efficiency loss that such an outcome entails. In fact, in cardinal terms, summing over all players' utilities (as expressed with a numeraire good), the social welfare of any JS-stable outcome is at least 50% of the most efficient outcome. We then show that the family of AFS production functions is the largest set of production functions for which our welfare theorems hold.

A shortcoming of our model is that it views the labor market as static. A static model admittedly cannot handle the following temporal argument: firms may be more cautious in hiring initially if they eventually face limitations on their ability to reduce the workforce. Such an argument is captured in a variety of general equilibrium labor economics models; we discuss specific references in Section 1.1. We believe that this argument does not make the study of a static solution vacuous. In fact, any rest point of a dynamic setting cannot be unstable, as it will clearly imply deviations and further deviations. In other words, we argue that the study of stability in the static case tells us what outcomes must be excluded even in the dynamic case. Thus, as Kelso and Crawford argue, we must understand the static case before developing the dynamic model, because stability may simply not exist, for example, due to the nature of the production functions, and regardless of the dynamics being considered. The current contribution should therefore be viewed as the first in a sequence of works intended to study job security in static and later dynamic matching models.

The notion of JS-stability is primarily motivated by regulatory intervention designed to increase job security in labor markets. However, it may also have relevance in the study of immigration and community formation. In this context, matching takes place between countries on the one hand and citizens on the other hand. Thus, firms are replaced by countries and workers by citizens. In such matching markets an asymmetric notion of stability is needed since citizenship, once granted, is almost impossible to revoke. On the other hand, although there exists a barrier for citizens to immigrate and replace their current citizenship with a different one such a barrier is clearly lower, which can be evidenced empirically. Thus, a variant of JS-stability to such a NTU setting may correctly represent the feasible community structure in a model of immigration.[2] In fact, there may be additional many-to-one matching markets where divorce costs on both sides of the market are highly asymmetric and so the notion of JS-stability becomes an adequate tool for their analysis.

The paper is organized as follows. Section 1.1 discusses the related literature. Section 2 introduces the model and details the new solution concept as well as the class of production functions we study. Section 3 provides the main results, and Section 4 gives concluding remarks. Some proofs are postponed to the Appendix.

1.1 Related Literature

The existence of stable outcomes under weaker notions of substitutability has received recent attention in the literature on matching with contracts, a model of labor markets and stability that originates with Hatfield and Milgrom (2005) and generalizes Kelso and Crawford (1982). Within this model Hatfield and Kojima (2010) define two notions, "bilateral substitutes" and "unilateral substitutes", that extend the original substitutes condition and still ensure existence of stable outcomes. Sönmez and Switzer (2013) and Sönmez (2013) demonstrate the applicability of these extended classes in the context of the "cadet-branch matching problem". However, these new classes shed no light on the original Kelso and Crawford (1982) model. When folding back the new classes of production function into the Kelso and Crawford (1982) model one obtains that the classes of "bilateral substitutes", "unilateral substitutes" and "gross substitutes" are one and the same.[3] This is no surprise given the maximality theorem of Gul and Stacchetti (1999) which argues that one cannot go beyond the class of gross substitutes without considering weaker notions of stability as we do.

[2] We thank Yoram Weiss for pointing out this connection between JS-stability and community formation.

[3] The way to embed the latter model in the former is by restricting attention to contracts of the form of a triplet (m, n, s), interpreted as a contract where worker m is employed by firm n for the salary s.

A sequence of papers starting with Compte and Jehiel (2008), and more recently Pereyra (2013), study a dynamic two-sided matching model of labor markets with existing workers who are guaranteed to be matched with at least as good partners as their current ones. Although these papers share a similar motivation to ours, they have very different models: they considered matchings with non-transferable utilities (i.e., non-negotiable salaries), and their markets have specific sets of workers (i.e., existing workers) who have secured jobs. Additional papers in this strand are Kurino (2011) in the context of on-campus housing for college students (where freshmen apply to move in and graduating seniors leave) and Ünver (2010) in the context of kidney exchange. These papers focus on the unit demand case (one-to-one matching) and utilities are non-transferable.

The lion's share of the theoretical literature on job security and employment protection legislation makes use of partial and general equilibrium in dynamic models. A common thread of all these models is that the work force is assumed *homogeneous* (e.g., Gavin, 1986, Lazear, 1990, Acemoglu and Shimer, 2000, Bertola, 2004), which is in sharp contrast with our heterogeneity assumption. Typically in these papers, a firm's productivity depends on the size of the workforce but not on the exact composition of workers it employs. Whereas our model is static, these models are dynamic and information stochastically unravels with time (e.g., workers' productivity and firms' technology). Whereas our work is more concerned with existence and efficiency of stable outcomes with regulation, their focus is on the impact of regulation on unemployment rates. Interestingly, the findings of this literature, both theoretically and empirically, are inconclusive; see the survey by Bertola (1999). Although the current paper does not discuss unemployment rates we argue for the relevance of the new notion of stability to such an analysis. In particular, comparison of unemployment rates in stable versus JS-stable outcomes may shed light on this important topic.

2 Preliminaries

A labor market is composed of a finite set of firms and workers such that each firm hires as many workers as it wishes, but each worker is allowed to work only at one firm. Each firm pays its workers a salary and the utility of each worker depends on which firm he works for and the salary he receives. Each firm's objective function is its profit, defined as the difference between the value of its production (in salary units) and the salaries it pays out. Note, in particular, there are no externalities among workers nor among firms.

The formal model we use is due to Kelso and Crawford (1982). A labor market is a tuple (N, M, v, b) where N is a finite set of firms and M is a finite set of workers with quasi-linear utility functions; in the sequel we abuse notation and use N and M to denote the cardinality of these sets as well. In the tuple $v = \{v^n\}_{n \in N}$, $v^n : 2^M \to \Re_+$ denotes firm n's monotonically increasing[4] production function, as measured in the same units as salaries. We calibrate $v^n(\emptyset) = 0$. In the

[4] v^n is monotonically increasing if $C \subset D \implies v^n(C) \leq v^n(D)$.

tuple $b = \{b_m^n\}_{m \in M, n \in N}$, $-b_m^n$ denotes the valuation, in salary terms, of worker m for working at firm n without being paid. We typically think of b_m^n as the minimal salary required by worker m for working at firm n and hence the negation sign. Thus, the quasi-linear utility for this worker is $u_m(n, s) = s - b_m^n$ when her salary is s.[5] Hereinafter firm 0 will denote unemployed workers and we calibrate $b_m^0 = 0$ for all m. We refer to worker m as *salary-driven* if $b_m^n = 0$ for all n.

As productivity is measured in salary units, the profit of firm n from employing a set of workers C when workers' salaries are $\{s_m\}_{m \in M}$ is $\Pi^n(C; s) = v^n(C) - \sum_{m \in C} s_m$. We often abbreviate the tuple (N, M, v, b) to (v, b) as the sets of workers and firms are implicitly encoded in (v, b).

For any two disjoint sets of employees, C and D, we denote by $v(D|C) = v(D \cup C) - v(C)$ the marginal productivity of D given C and we also abuse notation and write m to denote the singleton set $\{m\}$ as well (hence $v^n(m)$ will denote the productivity of a single worker, m).

Our results require the following assumption that relates each worker's minimal salaries to his marginal productivity. This "marginal productivity assumption" (MP), originally made by Kelso and Crawford (1982), states that the marginal productivity of any firm from any employee is at least the employee's minimal desired salary. Formally,

$$\forall n,\ C \subset M,\ m \in M \setminus C, \quad v^n(m|C) \geq b_m^n. \tag{MP}$$

Notice that MP trivially holds for any salary-driven worker, i.e., a worker with all minimal salaries being equal to zero. More generally, we view this as a behavioral assumption on the way workers set minimal salaries. In particular, Kelso and Crawford (1982) justify this assumption by writing *"This is a natural restriction, since if a worker's marginal product, net of the salary required to compensate him or her for the disutility of work at a given firm, were negative, the firm could agree to let the worker do nothing for a salary of zero."* (section 2, page 1486).

An *assignment* of workers is a partition $A = \{A^0, A^1, \ldots, A^N\}$ of the set of workers, where A^n denotes all workers employed by firm n, with A^0 interpreted as the set of unemployed workers. An *allocation* is a pair (A, s) where A is an assignment of workers and $s \in \Re_+^M$ is a vector of salaries. Such an allocation implies that any employee $m \in A^n$ works for firm n at a salary s_m whenever $n > 0$, and $m \in A^0$ implies that m is unemployed and receives no salary.

Definition 1. An allocation (A, s) is *individually rational (IR)* if (1) $v^n(A^n) - \sum_{m \in A^n} s_m \geq 0\ \forall n \in N$; and (2) $s_m \geq b_m^n$ for all $n \in N$ and $m \in A^n$.

The first part of this definition requires that each firm has a non-negative net profit and the second part requires that each employed worker is paid her minimal required salary.

[5] The model and results in Kelso and Crawford (1982) make use of an abstract utility function for workers, not necessarily of a quasi-linear form. In particular the units of such functions are abstract utilities in contrast with our quasi-linear functions whose units are in salary terms. Thus, as opposed to the Kelso-Crawford model, we can discuss a cardinal measure of social welfare and consequently measure efficiency levels, which is central to our results. However, we do so without ignoring non-salary related components of the work-package as these are embedded in the minimal salary component (b in our model) which is dependent on the specific worker and the specific firm.

2.1 Stability and Job Security

The central solution concept we adopt is that of stability. However our notion of stability is a central innovation of our work and is weaker than the standard stability notions in two-sided markets. The stability notion we introduce is inspired by markets where job security is guaranteed by regulatory means. In particular, we consider the following simple yet somewhat extreme assertion: once a worker is employed by a firm for a certain salary, only the worker can decide to quit whereas the firm cannot lower the salary nor can it fire the worker.

The stability notion we introduce is an adaptation of the standard notion of stability to such regulatory restrictions. We now turn to define the new notion of stability in steps. First, recall the classic stability notion:

Definition 2. A coalition $\{n, C\}$ is a **blocking** coalition for an allocation (A, s) if and only if there exists a vector of salaries, $\hat{s} \in \Re_+^C$, such that:

1. $u_m(n, \hat{s}_m) \geq u_m(k, s_m) \ \ \forall k \in N, m \in A^k \cap C$ (workers in C are better-off),
2. $v^n(C) - \sum_{m \in C} \hat{s}_m \geq v^n(A^n) - \sum_{m \in A^n} s_m$ (firm n is better-off),

with at least one of the inequalities being strict. An allocation (A, s) is **stable** if and only if it is IR and there exist no blocking coalitions for it.

Our definition of a JS-blocking coalition is in the spirit of the above definition, adding to it the requirement that JS-blocking coalitions must contain all previosuly employed workers. In other words, JS-blocking coalitions are blocking coalitions in which the deviating firm is restricted to adding workers. This is done by adding a requirement that $A^n \subset C$. This is in fact the only difference between the two definitions.

Definition 3. A coalition $\{n, C\}$ is a **JS-blocking** coalition for an allocation (A, s) if and only if it is a blocking coalition, and additionally $A^n \subset C$. An allocation (A, s) is **JS-stable** if and only if it is IR and there exist no JS-blocking coalitions for it.

In words, the requirement for JS-stability, beyond IR, is that there exists no firm and no set of workers currently not working for this firm such that the firm can offer better working terms for these new workers (first requirement) while maintaining its current set of workers at their current salaries and increasing its profits (second requirement). This is a weaker notion than the core allocation defined by Kelso and Crawford (1982). While Kelso and Crawford require that an allocation be immune to a deviation by a coalition of workers and a firm where such workers may (partly) replace the firm's current work force, our notion ignores this possibility as it is banned by regulation.

Requiring individual rationality as part of the definition of JS-stability implies that a firm that is not profitable, and thus in danger of bankruptcy, need not comply with job security regulations. In other words, we only force *profitable* firms to comply with job security regulations.

One can naturally entertain the possibility that instead of firing employees a firm can induce them to voluntarily quit by making a sufficiently large buyout offer.[6] Seemingly, this possibility is not captured by the notion of JS-stability yet conforms with the regulatory environment we model. However, Lemma 4 in the Appendix shows that without loss of generality we do not need to consider JS-blocking coalitions involving such payments. The reason is that a firm always profits more by actually employing workers than by paying them to quit.

JS-stability models an extreme version of regulation related to job security. Thus, the inefficiency induced under JS-stability in the worst case may be seen as a lower bound on the efficiency implications of some more realistic regulation. Indeed, as we demonstrate in this work, in spite of our modeling choice, efficiency partly prevails. This suggests that weaker forms of regulation designed for job security do not necessarily contradict efficiency.

Technically, checking if a coalition is JS-blocking becomes simpler and more convenient using the following Lemma:

Lemma 1. *Let $C \subset M \setminus A^n$. The coalition $\{n, A^n \cup C\}$ is a JS-blocking coalition for the allocation (A, s) if and only if $v^n(C|A^n) > \sum_{m \in C} s_m - b_m^{k(m)} + b_m^n$, where for every $m \in C$, $k(m)$ is the firm that satisfies $m \in A^{k(m)}$.*

The proof of this lemma is straightforward and therefore omitted.

2.2 Efficiency

The *efficiency level* of an assignment A is $P_{(v,b)}(A) = \sum_n v^n(A^n) - \sum_{m \in A^n} b_m^n$; recall that $v^n(\cdot)$ and b_m^n are all measured in salary units. An assignment is *efficient* if it maximizes the efficiency, over all possible assignments. We sometimes drop the subscript (v, b) when it is clear from the context.

2.3 Fractionally subadditive production functions

It has long been recognized that the class of gross substitutes (GS) production functions captures only a restricted notion of substitutes. For example, GS does not even include all production functions that exhibit decreasing marginal productivity, i.e., GS is a strict subset of the class of all submodular (SM) production functions.[7]

The various structures that we will assume on the production technology significantly expand SM (let alone GS). A key ingredient in these structures is the class of fractionally subadditive (FS) production functions. The definition of this class uses the following notion: for any $C \subseteq M$, a vector of non-negative weights $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ is a *fractional cover* of C if for any $m \in C$,

[6] We thank Regis Renault for posing this question.

[7] In fact, Lehmann et al. (2006) argue that SM significantly expands GS, in the following sense. A production function in SM can be represented by a vector in $(2^M - 1)$-dimensional Euclidean space that specifies the value of the production function on every non-empty set. Under this natural representation, Lehmann et al. (2006) prove that the set *GS* has Lebesgue measure zero in the space *SM*.

$\sum_{\{D\subseteq C: m\in D\}} \lambda_D = 1$. An example of a fractional cover of the set $\{a,b,c\}$ is $\lambda_D = \frac{1}{2}$ for any subset with two workers and $\lambda_D = 0$ otherwise.

Definition 4. A firm's production function v is *fractionally subadditive on* $C \subseteq M$ if for any fractional cover $\{\lambda_D\}_{D\subseteq C, D\neq\emptyset}$ of C, $v(C) \leq \sum_{D\subseteq C, D\neq\emptyset} \lambda_D v(D)$.

We can offer the following intuition for this notion: assume a firm can either make use of the set C of workers during a single period or it can break C into subsets of workers (possibly overlapping) and deploy the subsets sequentially, each for a fraction of a period, such that any employee works one full period of time. The production function is fractionally subadditive on C if the latter option is always at least as productive as the former. In the example of a fractional cover preceding Definition 4, the firm will (weakly) prefer having the three workers work in three shifts of pairs, each for half of a time period, over employing all three workers simultaneously for a single time period.

Definition 5. A firm's production function v is *fractionally subadditive*, denoted $v \in FS$, if for any $C \subseteq M$, v is fractionally subadditive on C.

FS continues to enforce substitutability, as the above intuition suggests. In particular, for any $v \in FS$ and any two sets S, T, $v(S \cup T) \leq v(S) + v(T \setminus S) \leq v(S) + v(T)$, as the weights $\lambda_S = 1$ and $\lambda_{T\setminus S} = 1$ are a fractional cover of $S \cup T$. *FS* was defined by Nisan (2000) and by Feige (2009) in the context of combinatorial auctions. Lehmann et al. (2006) show that $SM \subset FS$.[8] Dobzinski et al. (2010) describe the following useful characterization of FS, that we refer to as "supporting salary vectors". As we describe in the sequel, our analysis heavily relies on this characterization.

Definition 6. A vector of salaries, s, is called a *supporting salary vector* for the production function v and a subset of workers $C \subset M$ if (1) $\sum_{m\in C} s_m = v(C)$; and (2) For any $D \subset C$, $\sum_{m\in D} s_m \leq v(D)$.

Theorem 1 (Dobzinski et al. (2010))**.** *A production function v is fractionally subadditive on $C \subseteq M$ if and only if there exists a non-negative supporting vector of salaries for v on C.*

As an aside, we remark that this is in fact the Bondareva-Shapley theorem. Specifically, consider the cooperative game where the set of players is the set of workers and the characteristic function is the production function. Then, fractional covers are exactly "balanced collections of weights", fractional subadditivity exactly corresponds to antibalancedness, and the set of supporting salary vectors for the grand coalition is exactly the anticore of the cooperative game. Theorem 1 thus becomes the Bondareva-Shapley Theorem.[9]

[8]Lehmann et al. (2006) give the following example to demonstrate that the inclusion is strict. Consider the following symmetric production function on three workers: any set of one or two workers produces 2, while the set of all three workers produces 3. This is clearly not in SM, and it can be easily verified that it belongs to FS.

[9]The notions of antibalanced functions and the anticore are mirror images of the notions of balanced functions and the core, where the inequalities in the definitions are reversed. The Bondareva-Shapley theorem links the core with balanced charateristic functions, and the anticore with antibalanced charateristic functions.

3 Results

We now describe our three main results. The first two connect JS-stability with efficiency and can be viewed as analogs for the First and Second Welfare Theorems. In particular, we define a class of production functions called AFS, and show that whenever production functions belong to this class, efficient outcomes are JS-stable. Our third result shows that AFS is the maximal such class. These results parallel central results in the literature on stability in labor markets (without job security). We informally summarize our results and their parallels toward the end of this section.

3.1 A $\frac{1}{2}$-First Welfare Theorem

As one can expect, JS-stability does not guarantee efficiency. On the other hand, the inefficiency of any JS-stable outcome is bounded.

Theorem 2. *If (A, s) is JS-stable and $\bar{A}$ is efficient, then $P(A) \geq \frac{1}{2}P(\bar{A})$.*

Note that this result is not restricted to any specific class of production functions; it holds for arbitrary monotone production technologies that satisfy the MP assumption. We sketch the proof here and supply a formal proof in the Appendix.

First consider the case of labor markets with salary-driven workers. The efficiency is then the sum of firms' productivities, $P(A) = \sum_{n \in N} v^n(A^n)$. JS-stability of (A, s) implies that for any firm n, the productivity gain from hiring all of the workers in $\bar{A}^n$ who do not already belong to A^n cannot exceed the sum of salaries of those workers. Summing over all firms, we may conclude that

$$P(\bar{A}) - P(A) \leq \sum_{m \in M} s_m.$$

On the other hand, individual rationality implies that the sum of salaries is bounded above by the sum of firms' productivities,

$$\sum_{m \in M} s_m \leq P(A).$$

Combining these two inequalities, we obtain $P(\bar{A}) - P(A) \leq P(A)$, which implies the bound stated in the theorem. Finally, to remove the assumption of salary-driven workers, we make use Lemma 6 in the Appendix which expresses several useful relationships between general labor markets and those with salary-driven workers.

The bound on the efficiency loss in Theorem 2 is tight, as the following example illustrates.

Example 1. Consider a labor market with two salary-driven workers a, b and two firms with unit-demand[10] production functions v^1, v^2, defined as follows: $v^1(a) = v^2(b) = 2$, $v^1(b) = v^2(a) = 1$.

[10] A unit-demand production function satisfies $v(B) = \max\{v(x) : x \in B\}$ for any $B \subseteq M$. In words, a coalition can only produce as much as its top producing member. These production functions are in GS, hence also in AFS.

The following allocation is JS-stable: firm 1 is matched to worker b, firm 2 is matched to worker a, and both salaries are 1. This allocation has welfare 2, while the efficient allocation has welfare 4.

The following example demonstrates that in the absence of the MP assumption, the inefficiency is potentially unbounded.

Example 2. Consider a market with two workers and one unit demand firm, who values each worker for 2. Each worker has a minimal salary of 1. Notice that MP is violated in this setting, since the marginal production of each worker, given the other worker, is zero, while his minimal salary is 1. Assigning both workers to the firm, with a salary of 1 to each worker, is a JS-stable outcome. This outcome has zero welfare, while the optimal welfare is 1.

3.2 A Second Welfare Theorem

Our first analog of the second welfare theorem guarantees the existence of an efficient JS-stable allocation for any set of production functions in FS. This theorem will later be generalized to the slightly broader class of AFS production functions in the next subsection.

Theorem 3. *Let (v, b) be a labor market. If $v^n \in FS$ for all $n \in N$, then for any efficient assignment A there is a salary vector s such that (A, s) is a JS-stable allocation.*

Given an efficient assignment, the proof shows that setting salaries to be supporting salary vectors for the assignment yields a JS-stable allocation. Since production functions are in FS, such salary vectors are guaranteed to exist.

Proof. We prove the claim for a salary-driven labor market. The proof for an arbitrary labor market follows from Lemmas 5 and 6 in the Appendix.

Let $A = (A^1, \cdots, A^n)$ be some efficient assignment. Theorem 1 implies that for each $k \in N$ there exists a supporting vector of salaries $\{s_m^k\}_{m \in A^k}$, for (v^k, A^k). For any $m \in M$ let $n(m)$ denote the firm for which $m \in A^{n(m)}$ and set $s_m = s_m^{n(m)}$. We show that the allocation (A, s) is JS-stable. IR follows immediately from the definition of a supporting vector of salaries. We show that an arbitrary coalition, (n, B), where $B \subset M \setminus A^n$, cannot be a blocking coalition. Denote $R^k = A^k \cap B$. As A is efficient $v^n(A^n \cup B) + \sum_{k \neq n} v^k(A^k \setminus R^k) \leq \sum_{k \in N} v^k(A^k)$. Therefore $v^n(A^n) + v^n(B|A^n) \leq \sum_{k \in N} v^k(A^k) - \sum_{k \neq n} v^k(A^k \setminus R^k) = v^n(A^n) + \sum_{k \neq n} v^k(R^k|A^k \setminus R^k)$. As $\{s_m^k\}_{m \in A^k}$ is a vector of supporting salaries for (v^k, A^k) we have $v^n(B|A^n) \leq \sum_{k \neq n} v^k(R^k|A^k \setminus R^k) \leq \sum_{k \neq n} \sum_{m \in R^k} s_m^k = \sum_{m \in B} s_m$, implying that (n, B) is not a blocking coalition. ☐

3.3 On JS-stability and maximal sets of production functions

In the previous section we observed that if all production functions are in FS then the existence of JS-stable allocations is guaranteed. In fact it was shown that any efficient assignment can be supported by a JS-stable allocation. A natural question now presents itself: is the class of FS production functions maximal for these observations to hold? That is, can one go beyond FS and still guarantee the existence of JS-stable allocations or even the existence of efficient JS-stable allocations?

In this section we define a new class of valuations, AFS, that strictly contains FS, and that supports efficient allocations by JS-stable salaries. We will motivate its definition by way of considering a couple of seemingly special scenarios. We will then show that AFS is in fact the maximal class capable of supporting efficient allocations as JS-stable outcomes.

We start by highlighting one specific requirement (out of the many FS requirements) that turns to be out essential to guaranteeing the existence of JS-stable outcomes. Specifically, this is the requirement $v(M) \leq \frac{1}{|M|-1} \sum_{x \in M} v(M \setminus x)$. Lemma 8 in the Appendix shows that if this requirement is violated, we cannot guarantee existence of JS-stable outcomes. This observation motivates the following definition:

Definition 7. A valuation u is called *symmetrically fractionally subadditive* if for any $B \subseteq M$ with $|B| \geq 2$,

$$u(B) \leq \frac{1}{|B|-1} \sum_{x \in B} u(B \setminus x).$$

Let *SFS* denote the set of all symmetric fractionally subadditive functions.

Note that for any subset of workers B, the collection of subsets $(B \setminus x)_{x \in B}$ is a fractional cover of B using uniform weights $\frac{1}{|B|-1}$. Thus, any $u \in FS$ must satisfy the required inequality in the definition of SFS. In other words, $FS \subset SFS$.

We noted above that if the SFS requirement corresponding to $B = M$ is violated, JS-stability cannot be guaranteed. It turns out that this generalizes to any SFS requirement. If a single SFS requirement is violated, JS-stability cannot be guaranteed. (See Proposition 2.)

A partial complement to these claims is also true: if all production functions are in SFS, and there exists a single firm which is by far more productive than all other firms in the sense that assigning all workers to this firm is an efficient assignment (hereinafter we refer to such a firm as a *superior* firm), then an efficient JS-stable allocation exists.

Proposition 1. *Let (v, b) be a labor market and assume firm n is superior. If $v^n \in SFS$ then assigning all workers to firm n can be supported as a JS-stable allocation.*

Proof. We prove the claim for a salary-driven labor market. As before, the proof for an arbitrary labor market follows from Lemmas 5 and 6 in the Appendix.

Let A denote the assignment of all workers to n. A is efficient (as n is superior) hence for any $k \neq n$ and any $m \in M$, $v^k(m) + v^n(M \setminus m) \leq v^n(M) = v^n(m|M \setminus m) + v^n(M \setminus m)$. Therefore, $v^k(m) \leq v^n(m|M \setminus m)$. Now, set $s_m = v^n(m|M \setminus m)$ for every $m \in M$. We show that this salary vector yields a JS-stable allocation. By Lemma 7 in the Appendix,

- $v^n(M) \geq \sum_{m \in M} v^n(m|M \setminus m) = \sum_{m \in M} s_m$, implying IR.
- $\forall k \neq n$ and $B \subseteq M$, $v^k(B) \leq \sum_{m \in B} v^k(m) \leq \sum_{m \in B} v^n(m|M \setminus m) = \sum_{m \in B} s_m$, implying that there exist no blocking coalitions.

$\square$

This proof establishes a second scheme to set salaries in JS-stable allocations. Recall that the salary vector required for supporting an efficient assignment as a JS-stable allocation derived as in Theorem 3 uses the full power of fractional subadditivity. The last proof has shown that devising salaries when all workers are assigned to a superior firm requires less, namely that production functions are in SFS (but not necessarily in FS).

On the other hand, if no superior firm exists then requiring all production functions to be in SFS does not guarantee the existence of an efficient JS-stable allocation, as we now demonstrate.

Example 3. There are four salary-driven workers $M = \{a, b, c, d\}$ and two firms $N = \{1, 2\}$ with corresponding production functions v^1, v^2 . The following table provides the production level for nonempty subsets of $\{a, b, c\}$:

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
v^1	5	3	3	6	6	6	9
v^2	3	3	3	6	6	4	8

We additionally have (1) $v^1(d) = 1.1$ and $\forall S \subseteq M \setminus \{d\}, S \neq \emptyset$, $v^1(d|S) = 0$, (2) $\forall S \subseteq M$, $v^2(d|S) = 0$. We leave it to the reader to verify that $v^1, v^2 \in SFS$. None of the firms is superior, and in the unique efficient assignment, $\{d\}$ works for firm 1 and $\{a, b, c\}$ work for firm 2. To see that this allocation cannot be made JS-stable, note that JS-stable salaries s for this allocation must satisfy $s_a \geq 3.9$ (since $v^1(a|d) = 3.9$) and $s_b + s_c \geq 4.9$ (since $v^1(bc|d) = 4.9$). However, we also need $s_a + s_b + s_c \leq v^2(abc) = 8$, a contradiction.

Theorem 3 and Proposition 1 describe two different schemes for devising salaries to support efficient assignments as JS-stable allocations, depending on the presence or absence of a superior firm. Combining these two schemes together gives the following class of production functions that is strictly larger than FS and that still guarantees the existence of efficient and JS-stable allocations.

Definition 8. A firm's production function v is *almost fractionally subadditive*, denoted $v \in AFS$, if:

1. For any $C \subset M$ (excluding $C = M$) v is fractionally subadditive on C, and
2. $v(M) \leq \frac{1}{|M|-1} \sum_{m \in M} v(M \setminus m)$.

The first requirement corresponds to the requirements for FS on strict subsets of M and hence is useful when an efficient assignment dictates that firms are assigned strict subsets of M. The second requirement corresponds to the SFS requirement and hence is useful when an efficient assignment assigns all workers to a single firm. Indeed, we have the following analog of the Second Welfare Theorem which extends our earlier Theorem 3.

Theorem 4. *Let (v, b) be a labor market. If $v^n \in AFS$ for all $n \in N$, then for any efficient assignment A there is a salary vector s such that (A, s) is a JS-stable allocation.*

Proof. If an efficient assignment A does not assign all workers to a single firm then the salary scheme s devised in the proof of Theorem 3 guarantees that (A, s) is JS-stable. If A does assign all workers to a single firm then this firm is superior and the result now follows from Proposition 1. □

It immediately follows from the definition that $FS \subset AFS \subset SFS$. Note that AFS allows for a certain type of complementarities: a single worker and the set of all other workers may be complements. This is because we do not require fractional subadditivity to hold on the full set of workers but only on strict subsets. The following example illustrates this complementarity.

Example 4. Assume there are 3 workers denoted a, b, c, and let the production function u be defined by: $u(a) = u(b) = u(c) = 3$, $u(\{a, b\}) = u(\{a, c\}) = 6$, $u(\{b, c\}) = 4$, $u(\{a, b, c\}) = 8$. We leave it to the reader to verify that $u \in AFS$ but not in FS. Note that the worker a and the pair $\{b, c\}$ are complements.

The techniques developed so far for proving that any efficient assignment can also be supported as a JS-stable allocation, as witnessed in the proof of Theorem 4, have motivated the notion of AFS. Thus, one may suspect that it may be possible to extend the proof beyond AFS. It turns out, quite surprisingly, that this is not the case and that the class of production functions AFS is a "natural" class in the context of efficient JS-stable allocations, in the sense that it is a maximal class with respect to aforementioned property. This is the content of our next result.

Theorem 5. *If $\bar{v} \notin AFS$ then there exists a labor market with salary-driven workers, in which one firm has production function $\bar{v}$ and the others have AFS production functions, such that there does not exist any JS-stable allocation (A, s) in which A is an efficient assignment.*

The proof is composed of two parts. We first show that if $v \notin SFS$, JS-stable allocations need not necessarily exist. We then show that if $v \in SFS \setminus AFS$, efficient JS-stable allocations need not necessarily exist.

Proposition 2. *If $v \notin SFS$ then there exist $k \leq 2n - 1$ unit-demand production functions $u_1, ..., u_k$ such that the salary-driven labor market with $k + 1$ firms, $((u_1, ..., u_k, v), 0)$, does not admit any JS-stable allocation.*

The Appendix provides the proof of Proposition 2 and of the following two lemmas, which establish properties of FS production functions that are needed in order to construct a production function $u^\epsilon \in FS$ such that the labor market $(\bar{v}, u^\epsilon)$ has no efficient JS-stable allocation.

Lemma 2. *For any valuation v and positive number r, let $v+r$ be the valuation defined as follows: $(v+r)(D) = v(D) + r$, for all $D \subseteq M$. For any monotone valuation v, there exists some positive number R such that for any $r \geq R$, $v + r \in FS$.*

Lemma 3. *For any $T \subset M$ let $v|_T(\cdot)$ denote the restriction of $v(\cdot)$ to T. If $v \notin FS$ and for some proper subset $T \subset M$, $v|_T \in FS$ then $v(T) < v(M)$.*

We are now ready to prove Theorem 5; Example 3 provides some intuition for this proof.

Proof of Theorem 5. If for some $B \subseteq M, \bar{v}(B) > \sum_{m\in B} \frac{\bar{v}(B\setminus m)}{|B|-1}$ then the conclusion follows from Proposition 2. Thus assume that for all $B \subset M$, $\bar{v}(B) \leq \sum_{m\in B} \frac{\bar{v}(B\setminus m)}{|B|-1}$. As $\bar{v} \notin AFS$ there exists some proper subset $T \subset M$ such that $\bar{v}|_T \notin FS$. In particular, let T be a minimal such subset. By Lemma 3, for any T' that is a proper subset of T, $\bar{v}(T') < \bar{v}(T)$. In particular we may choose $\bar{\epsilon} > 0$ be such that for any T' that is a proper subset of T, $\bar{v}(T') + \bar{\epsilon} < \bar{v}(T)$.

For any ϵ in the open interval $(0, \bar{\epsilon})$, we define the valuation u^ϵ on M as follows. Letting the notation D^c denote the complementary set of D, i.e. $D^c = M \setminus D$, we set $u^\epsilon(D) = r - \bar{v}(D^c)$ for all $D \neq T^c$ and $u^\epsilon(T^c) = r - \bar{v}(T) + \epsilon$, where $r = r(\epsilon)$ is large enough to guarantee that $u^\epsilon \in FS$. (Here we are applying Lemma 2.) Monotonicity of u^ϵ is straightforward from the construction and the choice of ϵ.

Allocating T to the firm with production function $\bar{v}$ and T^c to the firm with production function u^ϵ is the unique optimal allocation. Note that it generates an efficiency level of $r + \epsilon$ whereas any other allocation generates r.

Assume the theorem is false and that for any ϵ the unique optimal assignment of $(\bar{v}, u^\epsilon)$ can be supported by a JS-stable allocation $((T, T^c), s^\epsilon)$. By individual rationality, $\sum_{m\in T} s^\epsilon_m \leq \bar{v}(T)$. Increasing the salary of some single worker in T if necessary, we can assume without loss of generality that $\sum_{m\in T} s^\epsilon_m = \bar{v}(T)$. For any $D \subseteq T$, JS-stability implies

$$\sum_{m\in D} s^\epsilon_m \geq u^\epsilon(D|T^c) = u^\epsilon(D \cup T^c) - u^\epsilon(T^c) = \bar{v}(T) - \bar{v}(T \setminus D) - \epsilon = \sum_{m\in T} s^\epsilon_m - \bar{v}(T \setminus D) - \epsilon.$$

Therefore, for any $D \subseteq T$, $\sum_{m\in T\setminus D} s^\epsilon_m \leq \bar{v}(T \setminus D) + \epsilon$. This can be equivalently stated as follows:

$$\sum_{m\in D} s^\epsilon_m \leq \bar{v}(D) + \epsilon \quad \forall D \subseteq T.$$

Let $\epsilon_1, \epsilon_2, \ldots$ be a decreasing sequence in the open interval $(0, \bar{\epsilon})$ with $\lim_n \epsilon_n = 0$, and let s be an accumulation point of the set of salary vectors $\{s^{\epsilon_n}\}_{n=1}^\infty$. Then $\sum_{m\in T} s_m = \bar{v}(T)$ and

$\sum_{m\in D} s_m \le \bar{v}(D) \quad \forall D \subset T$ which implies that s is a supporting vector of salaries for $\bar{v}|_T$ on the set T, contradicting the assumption that $\bar{v}|_T \notin FS$. $\square$

The results of this section leave open the question of the maximal set of production functions that guarantee the existence of (possibly inefficient) JS-stable outcomes. In particular, we do not know whether such an allocation necessarily exists in *SFS* $\setminus$ *AFS*. We consider this to be a very interesting and technically challenging problem for future research. More specifically, one can easily verify that for three workers or less, the two classes *AFS* and *SFS* are the same. Hence, as a corollary of Theorem 4 and Proposition 2 we know that for markets with three workers or less, *AFS* is maximal with respect to the existence of JS-stable outcomes. With four or more workers, *SFS* strictly contains *AFS*, and so it is possible that whenever production functions are in $SFS \setminus AFS$, efficient JS-stable allocations do not exist but still JS-stable (inefficient) allocations are guaranteed to exist. This possibility is illustrated in the following example.

Example 5. Recall the market from Example 3. As argued $v^1, v^2 \in SFS$. On the other hand, $v^2(abc) = 8 > 3+4 = v^2(a)+v^2(bc)$ and so $v^2 \notin AFS$. Recall that no efficient JS-stable allocations exist. However, assigning all workers to firm 1 with salaries $s_a = s_b = s_c = 3$ and $s_d = 0$ is JS-stable yet inefficient.

3.4 Summary of Results

The following table informally summarizes our main results while comparing them with the existing literature on unregulated labor markets:

TYPE OF LABOR MARKET	UNREGULATED (existing literature)	REGULATED (our contribution)
Solution concept	Stable allocations	JS-stable allocations
Set of production function	GS	AFS and SFS
First welfare theorem	Stable allocations are efficient.	JS-stable allocations obtain half the maximal efficiency.
Second welfare theorem	Pareto efficient allocations are stable in GS.	Efficient allocations are JS-stable in AFS.
Maximality	Stable allocations are not guaranteed outside GS.	Efficient JS-stable allocations are not guaranteed outside AFS. JS-stable allocations are not guaranteed outside SFS.

4 Concluding Remarks

In this work we introduce JS-stability as a new solution concept for many-to-one matching markets. This concept is inspired by regulated labor markets where costs for firing employees are prohibitively high. We identify a large and maximal family of production functions which are not guaranteed to admit stable outcomes, yet JS-stable outcomes not only exist for these production functions but in fact support all efficient outcomes. While JS-stability does not always guarantee efficiency, it does guarantee at least 50% of the welfare of the first best outcome.

Our results can also shed light on markets with a single seller, many buyers and multiple goods (combinatorial auctions), where buyers correspond to firms (replacing production functions with valuation functions) and items correspond to workers. The assignment of workers to firms and the salaries of workers in a stable outcome correspond to the assignment of items to buyers and the prices of items in a Walrasian equilibrium. Our notion of job-security corresponds to an outcome where buyers do not want additional items on top of the items in their bundle, but may want to discard some of the items in their bundle. This solution concept may be useful in the context of governmental auctions in heavily regulated markets (e.g., FCC auctions) where an important concern of the designer is to set prices in a way that will eliminate a secondary market. Discarding items is less of an issue in such markets, since in these markets the seller is free to bundle the items. In such cases individual rationality will ensure that buyers accept the bundles offered to them.

Harnessing the many-to-one matching model for studying regulation in labor markets is novel, to the best of our knowledge. Thus, our work may be viewed as the first step of a research agenda that studies implications of regulatory intervention in labor markets. We highlight some natural follow-up questions which we leave for future research.

A major concern in labor theory is the effect of job security on unemployment rates. On the one hand, job security reduces layoffs, but on the other hand, at the hiring stage firms take job regulations into account and so tend to hire less. It seems interesting to compare employment levels in stable versus JS-stable outcomes, when both exist. In general, such comparative statics can swing both ways. The following example demonstrates that JS-stable outcomes might exhibit higher employment levels compared with a stable outcome of the same market.[11]

Example 6. There are two firms, A and B, and three workers a, b, c. Let $b_a^A = 1$ and $b_m^n = 0$ otherwise. Firm A has unit demand and $v^A(a) = 0, v^A(b) = 1, v^A(c) = 1.5$. For firm B, $v^B(a) = 4$, $v^B(b) = 6$, $v^B(c) = 2$ and $v^B(X) = 6$ for any set X of two or more workers. Note that matching b with B and c with A, both at zero salary, is a stable matching which leaves a unemployed. On the other hand matching b with A at a salary of 1, a and c with B at salaries 4 and 2, respectively, yields a JS-stable outcome with no unemployment. (Note that this matching is not efficient nor stable as B and b at a salary of 5 is a blocking coalition.)

[11]We thank Fuhito Kojima for suggesting this example.

Another recent trend in labor theory is to study the implications of a requirement for severance payments when firms lay off employees, e.g., as suggested in Blanchard and Tirole (2008). In fact, some countries, like Denmark, already implement such a policy (Andersen, 2012). It will be interesting to replace the notion of JS-stability with an alternative solution concept which models more moderate regulation than tenure within the framework of many-to-one matching models. The research agenda may well go beyond severance payments and study other regulatory means designed for job protection and job security such as insurance institutions.

Some of our results refer to a cardinal notion of efficiency. For this notion to make sense we require that all utilities, for firms and for workers, are given in the same "currency". As a result our model assumes that firms' and workers' utilities are given in terms of money. Whereas for firms this is natural (as we identify utility with profits), for workers this is a limitation. Therefore, a study of JS-stability is called for when workers' utility functions go beyond additive-separable functions. This is particularly important if one would like to account for uncertainty without assuming workers are necessarily risk neutral.

The solution concept we focus on, JS-stability, is based on the nonexistence of blocking coalitions composed of a single firm and some workers. However, a JS-stable allocation can conceivably allow for a situation in which several firms could shuffle their current joint set of workers and possibly recruit additional workers to obtain an outcome that is better for all involved. Such a possibility may imply that what we refer to as a JS-stable allocation may not necessarily be stable, even when job-security provisions are instated. Thus, a stronger definition of stability, in the spirit of the core of a cooperative game, may be called for. This definition is provided in an extended version of the paper, which appears online (Fu et al., 2015). The extended version also makes a connection between JS-stable outcomes and Nash equilibria of a game where workers are assigned to firms through simultaneous second price auctions.

Appendix

A Salary-driven workers

One primitive of our model is the existence of minimal salaries. This allows for heterogeneity in the workers' utility across firms. Given the separable additive nature of workers' utility, it is not too surprising that for the purpose of our results one can assume that, without loss of generality, such minimal wages are fixed at zero and, in fact, utilities are homogeneous. In this appendix we formalize and prove this intuition.

Lemma 4. *Let (N, M, v, b) be a labor market satisfying the marginal productivity assumption (MP) and let (A, s) be an arbitrary allocation. Then for any firm n, and any $Y \subset A^n$, $v(A^n) - \sum_{m \in A^n} s_m \geq v(A^n \setminus Y) - \sum_{m \in A^n \setminus Y} s_m - \sum_{m \in Y} \max(s_m - b_m^n, 0)$.*

Note that the left-hand side is the profit of firm n at the allocation (A, s). Also note that a firm must pay its employee m at least $\max(s_m - b^n_m, 0)$ in order to induce him to quit. Therefore, the right-hand side is the same firm's profit if it pays some workers (those in an arbitrary set $Y \subset A^n$) in order to quit.

Proof. From MP we get that $v(A^n) - v(A^n \setminus Y) \geq \sum_{m \in Y} b^n_m$. Subtracting $\sum_{m \in A^n} s_m$ on both sides gets us $v(A^n) - \sum_{m \in A^n} s_m \geq v(A^n \setminus Y) - \sum_{m \in A^n} s_m + \sum_{m \in Y} b^n_m$ where the right hand side is simply equal to $v(A^n \setminus Y) - \sum_{m \in A^n \setminus Y} s_m - \sum_{m \in Y}(s_m - b^n_m)$. This is larger than $v(A^n \setminus Y) - \sum_{m \in A^n \setminus Y} s_m - \sum_{m \in Y} \max(s_m - b^n_m, 0)$ and the result follows. $\square$

Definition 9. If (v, b) is some labor market, we denote by $(v - b, 0)$ a labor market with salary-driven workers and production functions $(v - b)^n(B) = v^n(B) - \sum_{m \in B} b^n_m$.[12] Similarly, if (A, s) is some allocation then $s - b$ is the following vector of salaries: if $m \in A^k$ then $(s - b)_m = s_m - b^k_m$.

Lemma 5. *Fix any $C \subset M$, any production function v, and any vector of minimal salaries b. Then, $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ is a fractional cover of v on C if and only if it is a fractional cover of $v - b$ on C. This immediately implies that $v \in FS$ if and only if $v - b \in FS$, $v \in AFS$ if and only if $v - b \in AFS$, and $v \in SFS$ if and only if $v - b \in SFS$.*

Proof. We prove the first direction. Let $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ be a fractional cover of v on C, and so $\sum_{m \in C} b_m = \sum_{D \subseteq C, D \neq \emptyset} \lambda_D \sum_{m \in D} b_m$. Consequently

$$(v - b)(C) = v(C) - \sum_{m \in C} b_m \leq \sum_{D \subseteq C, D \neq \emptyset} \lambda_D v(D) - \sum_{D \subseteq C, D \neq \emptyset} \lambda_D \sum_{m \in D} b_m = \sum_{D \subseteq C, D \neq \emptyset} \lambda_D (v - b)(D),$$

implying that $v - b$ is fractionally subadditive on C. The opposite direction of the proof is similar and hence omitted. $\square$

Lemma 6. *The labor markets (v, b) and $(v - b, 0)$ obey the following relations.*

- *$P_{(v,b)}(A) = P_{(v-b,0)}(A)$. In particular, A is an efficient assignment for $(v - b, 0)$ if and only if it is an efficient assignment for (v, b).*
- *(A, s) is IR for (v, b) if and only if $(A, s - b)$ is IR for $(v - b, 0)$.*
- *(n, C) is a blocking coalition for the allocation (A, s) (in the market (v, b)) if and only if it is a blocking coalition for the allocation $(A, s - b)$ in the labor market $(v - b, 0)$.*
- *(A, s) is a JS-stable in (v, b) if and only if $(A, s - b)$ is JS-stable $(v - b, 0)$.*

[12] The MP assumption implies that $(v - b)^n$ remains monotone.

Proof. The first statement follows using a straightforward calcualtion which is therefore omitted.

To prove the second statement, let (A, s) be an IR allocation for (v, b). Then, for each firm n, $v^n(A^n) \geq \sum_{m \in A^n} s_m$ which can be rewritten as $(v-b)^n(A^n) \geq \sum_{m \in A^n}(s_m - b^n_m) = \sum_{m \in A^n}(s-b)_m$. In addition, for each worker m, $s_m \geq b^n_m$, where $m \in A^n$. Equivalently , $(s-b)_m \geq 0$ which means that $(A, s-b)$ is IR for $(v-b, 0)$. The proof of the opposite direction is similar and hence omitted.

To prove the third statement, assume that (n, C) is a blocking coalition for (A, s) in the labor market (v, b). Then there exists some vector of salaries $\{\hat{s}_m\}_{m \in C}$ such that:

- $\hat{s}_m - b^n_m \geq s_m - b^k_m$ for all k and for all $m \in C \cap A^k$,
- $v^n(C|A^n) \geq \sum_{m \in C} \hat{s}_m$, implying $(v-b)^n(C|A^n) \geq \sum_{m \in C} \hat{s}_m - b^n_m$

with at least one of the inequalities being strict. Now set $\bar{s}_m = \hat{s}_m - b^n_m$ for all $m \in C$. The above system of inequalities is equivalent to:

- $\bar{s}_m \geq s_m - b^k_m = (s-b)_m$ for all k and for all $m \in C \cap A^k$,
- $(v-b)^n(C|A^n) \geq \sum_{m \in C} \bar{s}_m$,

with at least one of the inequalities being strict, implying the desired conclusion. The proof of the opposite direction is similar and hence omitted.

The fourth statement is a direct consequence of the previous two claims. $\square$

B Proof of Theorem 2

Theorem 2. *If (A, s) is JS-stable and $\bar{A}$ is efficient, then $P(A) \geq \frac{1}{2}P(\bar{A})$.*

Proof. We first prove our result for labor markets with salary-driven workers, denoted $(v, 0)$. Indeed, for every firm n we have $v^n(\bar{A}^n \setminus A^n | A^n) \leq \sum_{m \in \bar{A}^n \setminus A^n} s_m$. Thus, we have

$$v^n(\bar{A}^n) \leq v^n(\bar{A}^n \cup A^n) \leq \sum_{m \in \bar{A}^n \setminus A^n} s_m + v^n(A^n).$$

Therefore

$$\sum_{i=1}^{n} v^n(\bar{A}^n) \leq \sum_{i=1}^{n} \left(\sum_{m \in \bar{A}^n \setminus A^n} s_m + v^n(A^n) \right) \leq \sum_{i=1}^{n} \left(\sum_{m \in \bar{A}^n} s_m + v^n(A^n) \right)$$

$$\leq \sum_{m \in M} s_m + \sum_{i=1}^{n} v^n(A^n) = \sum_{i=1}^{n} \sum_{m \in A^n} s_m + \sum_{i=1}^{n} v^n(A^n) \leq 2 \sum_{i=1}^{n} v^n(A^n),$$

where the last inequality follows from (IR) of the assignment $A = (A^n)_{n \in N}$. This proves the claim for labor markets with salary-driven workers.

Now let (A, s) be a JS-stable allocation for an arbitrary labor market (v, b) and let $\bar{A}$ be an efficient assignment for (v, b). By Lemma 6, $(A, s - b)$ is a JS-stable allocation for $(v - b, 0)$ and $\bar{A}$ is efficient for $(v - b, 0)$. Now:

$$P_{(v,b)}(A) = P_{(v-b,0)}(A) \geq \frac{1}{2}P_{(v-b,0)}(\bar{A}) = \frac{1}{2}P_{(v,b)}(\bar{A}),$$

where the left and right equalities follow again from Lemma 6 and the inequality follows from the first part of the proof. □

C Proofs deferred from Section 3.3

The following lemma is required in the proof of Proposition 1.

Lemma 7. *If $u \in SFS$ then for all $B \subseteq M$,*

$$\sum_{m \in B} u(m|B \setminus m) \leq u(B) \leq \sum_{m \in B} u(m).$$

As a partial converse, if $\sum_{m \in B} u(m|B \setminus m) \leq u(B)$ for all $B \subseteq M$ then $u \in SFS$.

Proof. The left inequality is straightforward since

$$\sum_{m \in B} u(m|B \setminus m) = \sum_{m \in B} [u(B) - u(B \setminus m)] = |B| \cdot u(B) - \sum_{m \in B} u(B \setminus m).$$

The same calculation proves the partial converse.

To prove the right inequality we proceed by induction on $|M|$. The claim trivially holds for $|M| = 1$. For $|M| > 1$ and any B strictly contained in M we have the required property by the inductive assumption, since the restriction of u to the set of workers B is also a production function in SFS, and the inductive assumption holds for this production function. Thus, we only need to prove the property for $B = M$. Indeed,

$$u(M) \leq \frac{\sum_{m \in M} u(M \setminus m)}{|M| - 1} \leq \frac{\sum_{m \in M} \sum_{k \in M \setminus m} u(k)}{|M| - 1} = \sum_{k \in M} u(k),$$

as claimed. □

Lemma 8. *If a production function v has $v(M) > \frac{1}{|M|-1} \sum_{x \in M} v(M \setminus x)$, there exists a unit-demand valuation u such that the salary-driven market made up of two firms having production functions u and v does not have any JS-stable allocation.*

Proof. By rearranging $v(M) > \frac{1}{|M|-1} \sum_{x \in M} v(M \setminus x)$ we also have $\sum_{x \in M} v(x|M \setminus x) > v(M)$. Now define a unit-demand valuation u as follows. Choose a small enough $\epsilon > 0$ such that (i)

$\sum_{x \in M}(v(x|M \setminus x) - \epsilon) > v(M)$, and (ii) $\forall x \in M$ such that $v(x|M \setminus x) > 0$, $\epsilon < v(x|M \setminus x)$. Then define $u(x) = \max(0, v(x|M \setminus x) - \epsilon)$ for all $x \in M$.

We show that there does not exist a JS-stable allocation for the salary-driven labor market whose two firms have production functions u and v. Suppose towards a contradiction that there exists a JS-stable allocation (A, s). If $u(A^u) = 0$ (and thus $u(x) = 0$ for all $x \in A^u$), we have

$$\sum_{x \in A^v} s_x \leq v(A^v) \leq v(M) < \sum_{x \in M} (v(x|M \setminus x) - \epsilon) \leq \sum_{x \in A^v} u(x).$$

Thus, there exists a worker $x \in A^v$ with $s_x < u(x)$, and JS-stability is violated. Otherwise, $u(A^u) > 0$. Let $x^* = \arg\max_{x \in A^u} u(x)$, then $u(x^*) = v(x^*|M \setminus x^*) - \epsilon$. Since $\sum_{x \in M \setminus A^v} s_x = \sum_{x \in A^u} s_x \leq u(x^*)$ we have,

$$v(M \setminus A^v | A^v) - \sum_{x \in M \setminus A^v} s_x > v(M \setminus A^v | A^v) - v(x^*|M \setminus x^*) = v(M \setminus x^*) - v(A^v) \geq 0,$$

where the last inequality follows since $A^v \subseteq M \setminus x^*$. Once again this contradicts JS-stability. $\square$

Proposition 2. *If $v \notin SFS$ then there exist $k \leq 2n-1$ unit-demand production functions $u_1, ..., u_k$ such that the salary-driven labor market with $k+1$ firms, $((u_1, ..., u_k, v), 0)$, does not admit any JS-stable allocation.*

Proof. Since $v \notin SFS$, Lemma 7 implies that there exists $B \subseteq M$ such that $\sum_{x \in B} v(x|B \setminus x) > v(B)$. We construct the following tuple of unit-demand valuations. For every worker $x \in M \setminus B$ we have two unit-demand valuations $u_x^{(1)} = u_x^{(2)}$ such that $u_x^{(i)}(x) = v(M) + 1$ and $v_x^{(i)}(y) = 0$ for any worker $y \neq x$. Additionally let u_B be a unit-demand valuation defined similiarly to the unit-demand valuation in the proof of Lemma 8, namely $u_B(x) = \max(0, u(x|B \setminus x) - \epsilon)$ for all $x \in B$ and otherwise $u_B(x) = 0$. We argue that there does not exist a JS-stable allocation for this labor market. Note that in every possible JS-stable allocation in this labor market, every worker $x \in M \setminus B$ must be allocated to either the firm with valuation $v_x^{(1)}$ or $v_x^{(2)}$ and its salary must be $v_x^{(1)}(x)$. Thus, the set of workers assigned to either v or u_B is exactly B. The argument in Lemma 8 now shows that such an allocation cannot be JS-stable. $\square$

Lemma 2. *For any valuation v and positive number r, let $v + r$ be the valuation defined as follows: $(v + r)(D) = v(D) + r$, for all $D \subseteq M$. For any monotone valuation v, there exists some positive number R such that for any $r \geq R$, $v + r \in FS$.*

Proof. If $v(M) = 0$, then v is already in *FS*. Otherwise, let R be $(|M|-1)v(M)$, and we show for any $r \geq R$ that $v + r \in FS$, by constructing supporting salary vectors (Definition 6) for every $S \subseteq M$. For $S \subseteq M$ consider the vector $s \in \mathbb{R}^S$, where $s_x = \frac{r + v(S)}{|S|}$, for each $x \in S$. It is straightforward to see that $\sum_{x\in S} s_x = r + v(S) = (v+r)(S)$. Then for any proper subset $T \subsetneq S$,

$$(v+r)(T) \geq \sum_{x\in T} \frac{r}{|T|} = \sum_{x\in T} r \cdot \frac{|S|}{|T|} \cdot \frac{1}{|S|} \geq \sum_{x\in T} r\left(1 + \frac{1}{|M|-1}\right) \cdot \frac{1}{|S|} \geq \sum_{x\in T} \frac{r+v(S)}{|S|} = \sum_{x\in T} s_x.$$

In the second inequality we used the fact $\frac{|S|}{|T|} \geq \frac{|M|}{|M|-1}$, and in the last inequality we used the fact $r \geq R \geq (|M|-1)v(S)$. This shows that indeed s is a supporting salary vector, and therefore $v+r$ is in FS by Theorem 1. □

Lemma 3. *For any $T \subset M$ let $v|_T(\cdot)$ denote the restriction of $v(\cdot)$ to T. If $v \notin FS$ and for some proper subset $T \subset M$, $v|_T \in FS$ then $v(T) < v(M)$.*

Proof. For the sake of contradiction, suppose that for some $T \subsetneq M$, $v(T) = v(M)$ yet $v|_T \in FS$. Then there exists a supporting salary vector s on T. We extend s by setting $s_m = 0$ for all $m \in M \setminus T$, and we argue that we obtain a supporting salary vector for M, contradicting $v \notin FS$. To see this, observe that $\sum_{x\in M} s_x = \sum_{x\in T} s_x = v(T) = v(M)$, and for any $S \subsetneq M$, $\sum_{x\in S} s_x = \sum_{x\in S\cap T} s_x \leq v(S\cap T) \leq v(S)$. □

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